



4-1 Springs

Example 4-1 The McGraw-Hill Companies, Inc. Introduction to Mechanical Engineering, 4th Edition

4-3 Deflection Due to Bending Equations

$$q/EI = y^{(4)}$$

$$V/EI = y^{(3)}$$

$$M/EI = y^{(2)}$$

$$\theta = y'$$

$$y = f(x)$$

4-2 Deflection Due to Bending

Example 4-2 The McGraw-Hill Companies, Inc. Introduction to Mechanical Engineering, 4th Edition

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EXAMPLE 4-1 For the beam in Fig. 4-1, the bending moment at any cross, for $0 \leq x \leq L$, is

$$M = \frac{w_0}{2}x - \frac{w_0}{2}x^2$$

Using Eq. (4-12), determine the equations for the slope and deflection of the beam, the slopes at the ends, and the maximum deflection.

Solution Integrating Eq. (4-12) as an indefinite integral we have

$$\theta = \frac{d\delta}{dx} = \int M dx = \frac{w_0}{4}x^2 - \frac{w_0}{6}x^3 + C_1 \quad (1)$$

where C_1 is a constant of integration that is evaluated from geometric boundary conditions. We could suppose that the slope is zero at the midpoint of the beam, since the beam and

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loading are symmetric relative to the midpoint. However, we will use the given boundary conditions of the problem and verify that the slope is zero at the midpoint. Integrating Eq. (1) gives

$$\delta(x) = \int \theta dx = \frac{w_0}{12}x^3 - \frac{w_0}{24}x^4 + C_2x + C_3 \quad (2)$$

The boundary conditions for the simply supported beam are $\delta = 0$ at $x = 0$ and $x = L$. Applying the first condition, $\delta = 0$ at $x = 0$ in Eq. (2) results in $C_3 = 0$. Applying the second condition to Eq. (2) with $C_3 = 0$,

$$\delta(L) = \frac{w_0}{12}L^3 - \frac{w_0}{24}L^4 + C_2L = 0$$

Solving for C_2 and substituting $C_2 = -w_0L^2/24$ into Eq. (2) and simplifying the constants leads to Eq. (3) and solving for the deflection and slope results in

$$\delta = \frac{w_0}{24L^2}(12x^3 - x^4 - L^2x^2) \quad (3)$$

$$\theta = \frac{w_0}{24L}(12x^2 - 4x^3 - 2Lx) = 4x^2 - 2Lx \quad (4)$$

Comparing Eq. (3) with the given deflection curve shows that we've completed the problem. For the slopes at the left end, substituting $x = 0$ into Eq. (4) yields

$$\theta_{\text{left}} = -\frac{w_0L}{24L^2}$$

and at $x = L$,

$$\theta_{\text{right}} = \frac{w_0L}{24L^2}$$

At the midpoint, substituting $x = L/2$ into Eq. (4) gives $\theta = 0$, as we can expect. The maximum deflection occurs where $\theta = 0$. Substituting $x = L/2$ into Eq. (3) yields

$$\delta_{\text{max}} = -\frac{5w_0^2}{384L^2}$$

which again agrees with Table A-9-7.

Do Problem 4-12 on Board

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EXAMPLE 4-10 Determine the slope at point B and the deflection at point C for the beam in Fig. 4-10. Assume the distributed load varies linearly from 0 at A to 10 kN/m at B.

Solution First, draw the beam with the coordinate system and the load distribution. The load distribution is shown in Fig. 4-10a.

Integrating Eq. (4-12) as an indefinite integral we have

$$\theta = \frac{d\delta}{dx} = \int M dx = \frac{1}{2}wx^2 - \frac{1}{6}wx^3 + C_1 \quad (1)$$

$$\delta = \int \theta dx = \frac{1}{6}wx^3 - \frac{1}{24}wx^4 + C_2x + C_3 \quad (2)$$

where C_1 , C_2 , and C_3 are constants of integration that are evaluated from geometric boundary conditions. We could suppose that the slope is zero at the midpoint of the beam, since the beam and loading are symmetric relative to the midpoint. However, we will use the given boundary conditions of the problem and verify that the slope is zero at the midpoint. Integrating Eq. (1) gives

$$\delta(x) = \frac{1}{12}wx^3 - \frac{1}{24}wx^4 + C_2x + C_3 \quad (3)$$

The boundary conditions for the simply supported beam are $\delta = 0$ at $x = 0$ and $x = L$. Applying the first condition, $\delta = 0$ at $x = 0$ in Eq. (3) results in $C_3 = 0$. Applying the second condition to Eq. (3) with $C_3 = 0$,

$$\delta(L) = \frac{1}{12}wL^3 - \frac{1}{24}wL^4 + C_2L = 0$$

Solving for C_2 and substituting $C_2 = -wL^2/24$ into Eq. (3) and simplifying the constants leads to Eq. (4) and solving for the deflection and slope results in

$$\delta = \frac{w}{24L^2}(12x^3 - x^4 - L^2x^2) \quad (4)$$

$$\theta = \frac{w}{24L}(12x^2 - 4x^3 - 2Lx) = 4x^2 - 2Lx \quad (5)$$

Comparing Eq. (4) with the given deflection curve shows that we've completed the problem. For the slopes at the left end, substituting $x = 0$ into Eq. (5) yields

$$\theta_{\text{left}} = -\frac{wL}{24L^2}$$

and at $x = L$,

$$\theta_{\text{right}} = \frac{wL}{24L^2}$$

At the midpoint, substituting $x = L/2$ into Eq. (5) gives $\theta = 0$, as we can expect. The maximum deflection occurs where $\theta = 0$. Substituting $x = L/2$ into Eq. (4) yields

$$\delta_{\text{max}} = -\frac{5wL^2}{384L^2}$$

which again agrees with Table A-9-7.

4-8 Strain Energy Elements

(a) Pure shear element

(b) Beam bending element

Table 4-1
Strain-Energy Correction Factors for Shear

Source: Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999. Copyright © 1999 The McGraw-Hill Companies.

Beam Cross-Sectional Shape	Factor C
Rectangular	1.2
Circular	1.11
Thin-walled tubular, round	2.00
Box sections [†]	1.00
Structural sections [†]	1.00

[†]Use if available.

EXAMPLE 4-8 Find the strain energy due to shear in a rectangular cross-section beam, simply supported, and having a uniformly distributed load.

Solution Using Appendix Table A-4-1, we find the shear flow to be

$$V = \frac{w}{2} - vx$$

Substituting into Eq. (1-19), with $C = 1.2$, gives

Answer
$$U = \frac{1.2}{240} \int_0^L \left(\frac{w^2}{4} - wx \right) dx = \frac{w^2 L}{2040}$$

4-8 Example of Castigliano's Theorem

EXAMPLE 4-10 The cantilever of Fig. 4-10 is a uniform steel bar of length l with a free end fixed to a wall P or 200 kN . Find the maximum deflection using Castigliano's theorem, including the effect of shear. What error is introduced if shear is neglected?

Solution For Eqs. (1) and Example 4-8 find the total strain energy to

$$U = \frac{P^2 l}{240} + \int_0^l \frac{C V^2 dx}{240 E} \quad (1)$$

For the cantilever the shear force increases with position: $V = P$, then $C = 1.2$, from Table 4-1. Performing the integration and substituting these values into Eq. (1) gives the total strain energy.

$$U = \frac{P^2 l}{240} + \frac{1.2 P^2 l}{240 E} \quad (2)$$

Then, according to Castigliano's theorem, the deflection of the end is

$$\delta = \frac{\partial U}{\partial P} = \frac{P l}{240} + \frac{1.2 P l}{240 E} \quad (3)$$

Therefore the deflection is

$$\delta = \frac{P l}{240} + \frac{1.2 P l}{240 E} = 0.000417 P l$$

$$\delta = \frac{P l}{240} + \frac{1.2 P l}{240 E} = 0.000417 P l$$

ANSWER Deflection ignoring shear: $\delta = 0.000417 P l$. Deflection including shear: $\delta = 0.000417 P l + 0.000417 P l = 0.000834 P l$.

NOTE The error due to neglecting shear is 100%.