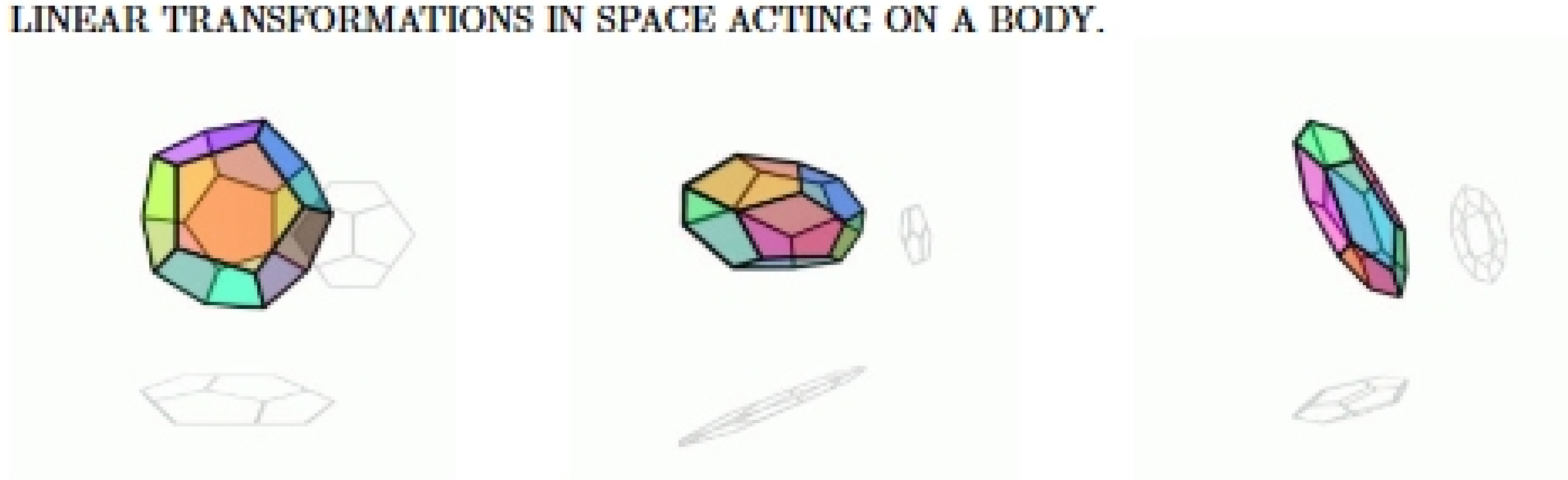
2/12/2002, LINEAR TRANSFORMATIONS (I) — Math 21b, O. Knill

HOMEWORK: due Thursday 2.1: 4,8*,10,20,28,42*, (Due next Tuesday: 2.2: 4,8,10,32,47*,50*, 2.3: 10,20,26*,30,40,42*

LINEAR TRANSFORMATION. A map T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is called a **linear transformation** if there is a $m \times n$ matrix with $T\vec{x} = \vec{y}$.

EXAMPLES. $T(\vec{x}) = \vec{y} \cdot \vec{x}$ from $\mathbf{R}^3$ to $\mathbf{R}$ ($A = \vec{y}$ is a 1×3 matrix (row vector). $T(x) = \vec{v}x$ from $\mathbf{R}$ to $\mathbf{R}^3$. $A = \vec{v}$ is a 3×1 matrix (column vector). $A\vec{x} = (x, y)$ from $\mathbf{R}^3$ to $\mathbf{R}^2$, A is a 2×3 matrix.

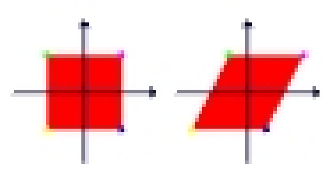
LINEAR TRANSFORMATIONS IN SPACE ACTING ON A BODY.



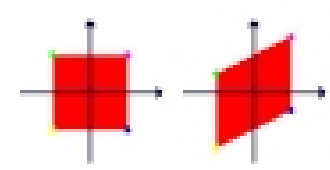
PROPERTIES OF LINEAR TRANSFORMATIONS. $T(\vec{0}) = \vec{0}$ $\quad T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ $\quad T(\lambda\vec{x}) = \lambda\vec{x}$
In words: Linear transformations are compatible with addition and scalar multiplication. It does not matter, whether we add two vectors before the transformation or add the transformed vectors.

SHEAR:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

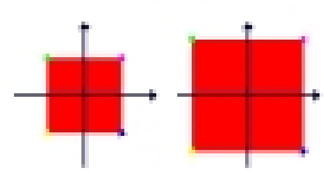


$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

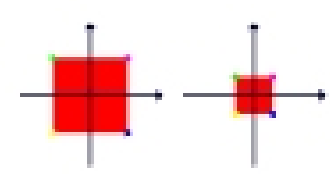


SCALING:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

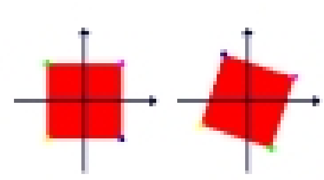


$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

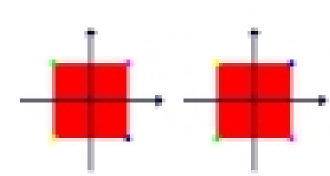


REFLECTION:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

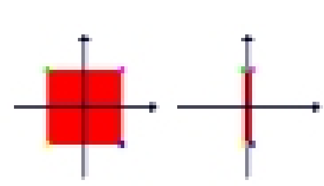


$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

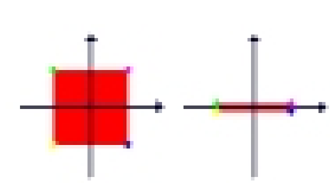


PROJECTION:

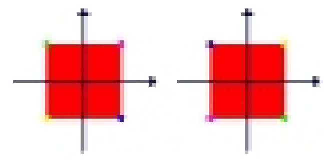
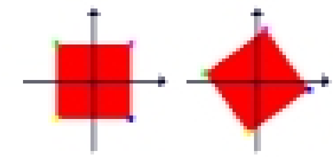
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



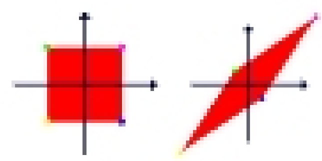
$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



ROTATION:

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$  $A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$ 

BOOST:

$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$ 

The boost is a basic **Lorentz transformation** in special relativity. It acts on vectors (x, ct), where t is time, c is the speed of light and x is space.

Unlike in **Galileo transformation** $(x, t) \mapsto (x + vt, t)$ (which is a shear), time t also changes during the transformation. The transformation has the effect that it changes length (Lorentz contraction). The angle α is related to v by $\tanh(\alpha) = v/c$. One can write also $A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)/\gamma x)$, with $\gamma = \sqrt{1 - v^2/c^2}$.

COLUMN VECTORS. A linear transformation $T(x) = Ax$ with $A = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix}$ has the property that the column vector $\vec{v}_1, \vec{v}_1, \vec{v}_n$ are the images of the **standard vectors** $\vec{e}_1 = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$. $\vec{e}_1 = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix}$. $\vec{e}_n = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$.

QUIZ. Find the linear transformation which rotates a cube around the diagonal $(1, 1, 1)$ by 120 degrees $(2\pi/3)$.

INVERSE OF A LINEAR TRANSFORMATION.
If S is a transformation such that $S(T\vec{x}) = \vec{x}$, for every $\vec{x}$, then S is called the **inverse** of T.

SOLVING A LINEAR SYSTEM OF EQUATIONS. $A\vec{x} = \vec{b}$ means to invert the linear transformation A. If the linear system has exactly one solution, then an inverse is possible. We write $\vec{x} = A^{-1}\vec{b}$. The inverse of the linear transformation is again a linear transformation.

ON LINEAR TRANSFORMATIONS. Linear transformations generalize scaling $x \mapsto ax$ in one dimensions. They are important in geometry, art (i.e. perspective, coordinate transformations), CAD applications (i.e. rotations, scales), physics (i.e. Lorentz transformations), dynamics (linearisations of general maps are linear maps), compression (Fourier transform), coding (many codes are linear codes).

LINEAR TRANSFORMATION OR NOT? (The square to the right is the image of the square to the left):