Problem Set 1

CSCE 222 Spring 2011

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Due on Friday, January 28, before class. Please use the cover sheet that will be available and staple your homework submission.

Let f, g be functions from the natural numbers $\mathbf{N} = \{1, 2, 3, \ldots\}$ to the set of real numbers $\mathbf{R}$. Recall that $f \preccurlyeq g$ means that that there exists an n_0 in $\mathbf{N}$ such that $f(n) \leq g(n)$ holds for all $n \geq n_0$.

Problem 1 Order the following five functions with respect to the relation $\preccurlyeq$:

$$n \ln n,\ (\ln \ln n)^{\ln n},\ ne^{\sqrt{n}},\ n^2,\ \ln n$$

Justify your answer.
[For example, since $1 \leq n$ holds for all $n \in \mathbf{N}$, multiplying both sides by $n > 0$ yields $n \leq n^2$ for all $n \geq n_0 := 1$. Therefore, $n \preccurlyeq n^2$ holds.]

Let $f, g \colon \mathbf{N} \to \mathbf{R}$. Recall that $f(n) = O(g(n))$ if and only if there exists a constant C such that $|f(n)| \preccurlyeq C|g(n)|$ holds.

Problem 2 Let f_1, f_2, g_1, g_2 be functions from $\mathbf{N}$ to $\mathbf{R}$, where $g_1(n) > 0$ and $g_2(n) > 0$ for all $n > 0$. Let $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$. Show that
(a) $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$
(b) $f_1(n)f_2(n) = O(g_1(n)g_2(n))$
holds. Do the claims also hold if g_1 and g_2 are not necessarily positive?

We write $f(n) = \Omega(g(n))$ if and only if $g(n) = O(f(n))$.

Problem 3 Let n be a natural number. We write $n!$ to denote the product $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. The number $n!$ is called n factorial and it counts for instance the number of ways to rearrange n distinct elements of an array. Show that
(a) $n! = O(n^n)$.
(b) $n! = \Omega(2^n)$.

We write $f(n) = \Theta(g(n))$ if and only if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold. Intuitively, this means that f and g have roughly the same asymptotic growth.

Problem 4 Show that $1^8 + 2^8 + 3^8 + \cdots + n^8 = \Theta(n^9)$.

Problem 5 Show that if c is a positive real number, then $g(n) = 1 + c + c^2 + \cdots + c^n$ satisfies
(a) $g(n) = \Theta(1)$ if $c < 1$,
(b) $g(n) = \Theta(n)$ if $c = 1$,
(c) $g(n) = \Theta(c^n)$ if $c > 1$.
Thus, the growth of the geometric series $g(n)$ is determined by its first coefficient if the series has decreasing terms, by the number of terms if the terms remain unchanged, and by the last coefficient if the terms are increasing. The geometric series is used very frequently in the analysis of algorithms.

Programming 1 *Install ruby on your computer. Explore the interpreter and learn about the basic data types (integers, arrays, hashes, strings, etc.) and control structures of ruby.*