More on Divisibility Proofs Name:______________________
A&S100, Section 002
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I think I have found a way to make the divisibilty tests for powers of 2 (i.e. $2, 4, 8, 16, 32$, etc.) a bit easier. (Yeah!) Before we get to that, let's look at a few examples. Consider the number $123,456$. We can break this number as follows:

$$\begin{aligned} 123,456 &= 123,000 + 456 \\ &= 123 \times 1000 + 456 \\ &= 123 \times 10^3 + 456 \end{aligned}$$

Or, if it suits our purposes better, we could break it up as

$$\begin{aligned} 123,456 &= 120,000 + 3,456 \\ &= 12 \times 10,000 + 3,456 \\ &= 12 \times 10^4 + 3,456 \end{aligned}$$

or even as

$$\begin{aligned} 123,456 &= 123,400 + 56 \\ &= 1234 \times 100 + 56 \\ &= 1234 \times 10^2 + 56 \end{aligned}$$

In fact, we can split this number anywhere we like in a similar manner. In the general case, we have

$$a_m \cdots a_4 a_3 a_2 a_1 a_0 = a_m \cdots a_4 a_3 a_2 a_1 \times 10 + a_0$$

or

$$a_m \cdots a_4 a_3 a_2 a_1 a_0 = a_m \cdots a_4 a_3 a_2 \times 10^2 + a_1 a_0$$

or

$$a_m \cdots a_4 a_3 a_2 a_1 a_0 = a_m \cdots a_4 a_3 \times 10^3 + a_2 a_1 a_0$$

or

$$a_m \cdots a_4 a_3 a_2 a_1 a_0 = a_m \cdots a_4 \times 10^4 + a_3 a_2 a_1 a_0$$

or, or, or, . . . Basically, we can keep this up until we run out of digits. With these general numbers we will not run out of digits. If you need an a_5, put it in. If you need an a_7, write the number as $a_m \cdots a_7 a_6 a_5 a_4 a_3 a_2 a_1 a_0$. We now have lots of different ways to write our number. We just need to figure out which one is most useful in the proof. In the case of proofs for divisibility by 2^k it happens to be the one that contains 10^k. Let's take a look.

1. Prove that a positive integer is divisible by 8 if and only if the number formed by the last three digits of the integer are divisible by 8.

   Note: $8 = 2^3$ *so we are going to break the number up at* 10^3.

   PROOF: Let $N = a_m \cdots a_3a_2a_1a_0$ be a positive integer. N is divisible by 8 if and only if

   $N \bmod 8 = 0$ if and only if

   $a_m \cdots a_3a_2a_1a_0 \bmod 8 = 0$ if and only if

   $(a_m \cdots a_3 \times 10^3 + a_2a_1a_0) \bmod 8 = 0$.

   Since $10^3 \bmod 8 = 0$, it now follows that $N \bmod 8 = 0$ if and only if

   $(a_m \cdots a_3 \times 0 + a_2a_1a_0) \bmod 8 = 0$ if and only if

   $a_2a_1a_0 \bmod 8 = 0$ if and only if

   the number formed by the last three digits of N is divisible by 8. □

2. Prove that a positive integer is divisible by 32 if and only if the number formed by the last five digits of the integer are divisible by 32.

   Note: $32 = 2^5$ *so we are going to break the number up at* 10^5.

   PROOF: Let $N = a_m \cdots a_5a_4a_3a_2a_1a_0$ be a positive integer. N is divisible by 32 if and only if

   $N \bmod 32 = 0$ if and only if

   $a_m \cdots a_5a_4a_3a_2a_1a_0 \bmod 32 = 0$ if and only if

   $(a_m \cdots a_5 \times 10^5 + a_4a_3a_2a_1a_0) \bmod 32 = 0$.

   Since $10^5 \bmod 32 = 0$, it now follows that $N \bmod 32 = 0$ if and only if

   $(a_m \cdots a_5 \times 0 + a_4a_3a_2a_1a_0) \bmod 32 = 0$ if and only if

   $a_4a_3a_2a_1a_0 \bmod 32 = 0$ if and only if

   the number formed by the last five digits of N is divisible by 32. □

3. Prove that a positive integer is divisible by 2 if and only if the last digit of the integer is divisible by 2. (*Or, said another way, the last digit is even.*)
Note: $2 = 2^1$ *so we are going to break the number up at* $10^1 = 10$.

PROOF: Let $N = a_m \cdots a_3 a_2 a_1 a_0$ be a positive integer. N is divisible by 2 if and only if

$N \bmod 2 = 0$ if and only if

$a_m \cdots a_3 a_2 a_1 a_0 \bmod 2 = 0$ if and only if

$(a_m \cdots a_3 a_2 a_1 \times 10^1 + a_0) \bmod 2 = 0$.

Since $10^1 \bmod 2 = 0$, it now follows that $N \bmod 2 = 0$ if and only if

$(a_m \cdots a_3 a_2 a_1 \times 0 + a_0) \bmod 2 = 0$ if and only if

$a_0 \bmod 2 = 0$ if and only if

the last digit of N is divisible by 2. $\square$

While we are discussing divisibility proofs, we might as well work through a couple more of them. The divisibility tests that follow are for divisors which are NOT powers of 2 so we can always go back to our old friend the base 10 expansion. Before doing that though, check to see if the new way of writing numbers will make things easier. Sometimes, it is helpful and can make for a shorter proof, as in the proofs for the divisibility tests by 5 and by 10. However, the new way of writing numbers does not appear to be helpful when trying to prove the divisibilty tests for 3 and 9 so we will probably have to continue to use the base 10 expanision when doing the proofs of the divisibility tests for 3 and 9. Sorry!