

Local Martingales and Quadratic Variation

Lecturer: Matthieu Cornec

Scribe: Brian Milch milch@cs.berkeley.edu

This lecture covers some of the technical background for the theory of stochastic integration. First, some notation: $M = (M_t)_{t \geq 0}$ is a process, and $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration. We assume \mathcal{F} is right-continuous and complete (\mathcal{F}_t includes the null sets for each t). If τ is a stopping time, then M^τ is M stopped at time τ :

$$M^\tau = (M_{t \wedge \tau})_{t \geq 0}$$

23.1 Local martingales

Definition 23.1 A process M is a local martingale w.r.t. \mathcal{F} if:

1. M is adapted to \mathcal{F} , that is, $\forall t M_t \in \mathcal{F}_t$
2. there exists a sequence (τ_n) of stopping times such that $\tau_n \uparrow \infty$ a.s., and M^{τ_n} is a true martingale for each n .

Definition 23.2 M is a local L^2 martingale if it satisfies Def. 23.1 with M^{τ_n} being an L^2 martingale for each n .

Other terms of the form “local $\langle adjective \rangle$ martingale (e.g., local bounded martingale) are defined similarly: we require that each M^{τ_n} be an $\langle adjective \rangle$ martingale. Note that “ $\langle adjective \rangle$ local martingale means something different: if we say that M is a bounded local martingale, we are saying that M is bounded and its a local martingale; were not saying anything special about the M^{τ_n} .

Remark 1: If M is a continuous local martingale, then we can take the M^{τ_n} to be bounded martingales. We can do this by letting $\tau_n = \inf\{t : |M_t| \geq n\}$; then since the paths are continuous, $|M^{\tau_n}| \leq n$.

Remark 2: Any continuous bounded local martingale is a true martingale. To see this, note that $M^{\tau_n} \uparrow M$, and since M is bounded we can apply the dominated convergence theorem.

Definition 23.3 Define the variation of M over the interval $[0, t]$ as:

$$V_t(\omega) = \sup_{\substack{n \in \mathbb{N} \\ 0 = t_0 < \dots < t_n = t}} \sum_{i=1}^n |M_{t_i}(\omega) - M_{t_{i-1}}(\omega)|$$

Then M has locally finite variation if $\forall t \exists C_t < \infty V_t < C_t$ everywhere.

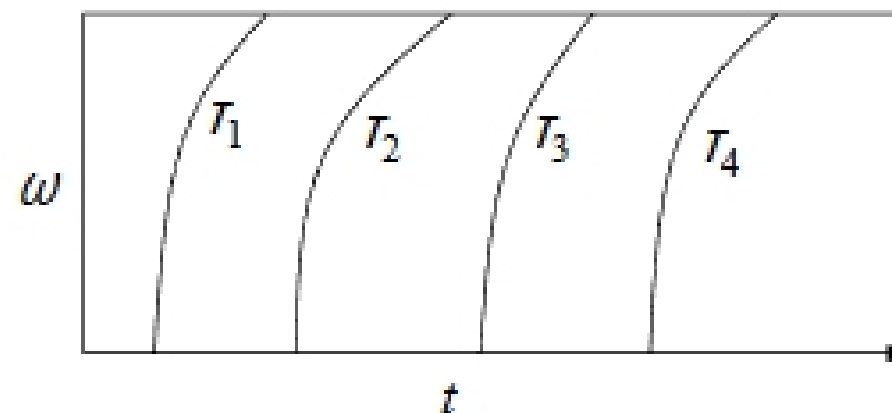


Figure 23.1: A localization argument involves proving a claim about a process X for those t and ω such that $t < \tau_n(\omega)$ — that is, those (t, ω) pairs to the left of the τ_n curve in this diagram — and then letting $n \rightarrow \infty$.

Proposition 23.1 (finite variation martingale) *If M is a continuous local martingale of locally finite variation then $M = M_0$ a.s.*

Proof: We can reduce this to the case where M is a true martingale with bounded variation and $M_0 = 0$ a.s. The reduction uses a localization argument: it suffices to show that $M^{\tau_n} = M_0$ a.s. for each n , and each M^{τ_n} is a true martingale. See the first paragraph of Kallenberg's proof (p. 330) for details, and Figure 23.1 for intuition.

Now for a fixed t and n , let $t_i = \frac{it}{n}$. Let:

$$\begin{aligned} \xi_n &\triangleq \sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2 \\ &\leq \left(\max_i |M_{t_i} - M_{t_{i-1}}| \right) V_t \\ &\rightarrow 0 \text{ a.s. since } V_t \text{ is bounded} \end{aligned}$$

Note that $\xi_n \leq V_t^2$ for all n and $EV_t^2 < \infty$, so the dominated convergence theorem implies $E\xi_n \rightarrow 0$ a.s. But:

$$\begin{aligned} E\xi_n &= E(M_t^2 - M_0^2) \text{ by orthogonality of martingale increments} \\ &= EM_t^2 \text{ because } EM_0^2 = 0 \end{aligned}$$

So $EM_t^2 = 0$, which implies $M_t = 0$ a.s.

We have proved this for arbitrary t , so we know $\forall t P(M_t = 0) = 1$. But we want to show $P(\forall t M_t = 0) = 1$. We do this by noting that $\forall t P(M_t = 0) = 1$ implies $P(\forall t \in \mathbb{Q}^+ M_t = 0) = 1$, and then concluding that $P(\forall t M_t = 0) = 1$ because M has continuous paths. ■

This proposition is used in proving the uniqueness of the covariation process (Thm. 23.5).

23.2 Stochastic integral of a step function

We want to define the stochastic integral $\int_0^t Y dX$, where (X_t) and (Y_t) are both processes, and $\left(\int_0^t Y dX\right)$ is another process. Kallenberg also uses the notation $(Y \cdot X)_t$ as a synonym for $\int_0^t Y dX$. The following definition handles the easy special case where Y is a step process:

Definition 23.4 Suppose we have a process X , stopping times $\tau_k \uparrow \infty$, random variables $\eta_k \in \mathcal{F}_{\tau_k}$, and a predictable step process:

$$V_t = \sum_k \eta_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t)$$

That is, V_t equals η_1 on $(\tau_1, \tau_2]$, η_2 on $(\tau_2, \tau_3]$, etc. Then the stochastic integral of the step process V with respect to X is:

$$(V \cdot X)_t = \sum_k \eta_k (X_{\tau_{k+1}}^t - X_{\tau_k}^t) \quad (23.1)$$

Recall that $X_{\tau_k}^t$ is the process X stopped at time t , and evaluated at time τ_k . So if $\tau_k > t$, then $X_{\tau_k}^t = X_t$.

One way to understand the stochastic integral is to imagine that $X_{\tau_{k+1}}^t - X_{\tau_k}^t$ is the fluctuation of a market between times τ_k and τ_{k+1} , and η_k is our “bet, or the number of shares in the market that we own between τ_k and τ_{k+1} . Then $\int_0^t V dX$ is the amount we gain in the market up to time t . By extending our definition of the stochastic integral to handle processes other than step processes, we will be able to model investment strategies that change the bet continuously.

Note that since $(V \cdot X)$ depends only on the changes in X :

$$(V \cdot X)_t = (V \cdot (X - X_0))_t$$

Recall that a martingale M is in L^2 if $\sup_t EM_t^2 < \infty$.

Proposition 23.2 For any continuous L^2 -martingale M where $M_0 = 0$, and any predictable step process V where $|V| \leq 1$, the process $(V \cdot M)$ is an L^2 -martingale with $E(V \cdot M)_t^2 \leq EM_t^2$.

Proof: First assume there are only a finite number of nonzero terms in V_t . We use the following lemma from Chapter 7 of Kallenberg:

Lemma 23.3 If M is a continuous martingale, τ is a stopping time, and $\zeta \in \mathcal{F}_\tau$, then the process $(N_t) = (\zeta(M_t - M_\tau^r))$ is also a martingale.

This process N_t is zero up to time τ , because for those times $M_t = M_\tau^r$. For $t \geq \tau$, $N_t = \zeta M_t - \zeta M_\tau$. We can rewrite the definition of $(V \cdot M)_t$ as a sum of processes of this form (see equation (1) in Chapter 17 of Kallenberg). So from the lemma and the assumption that the sum is finite, we can conclude that $(V \cdot M)_t$ is a martingale.

We still have to show $E(V \cdot M)_t^2 \leq EM_t^2$:

$$\begin{aligned} E(V \cdot M)_t^2 &= E \left(\sum_k \eta_k (M_{\tau_{k+1}}^t - M_{\tau_k}^t) \right)^2 \\ &= E \left(\sum_k \eta_k^2 (M_{\tau_{k+1}}^t - M_{\tau_k}^t)^2 \right) + 2E \left(\sum_{i < j} \eta_i \eta_j (M_{\tau_{j+1}}^t - M_{\tau_j}^t) (M_{\tau_{i+1}}^t - M_{\tau_i}^t) \right) \end{aligned}$$