Stat210B: Theoretical Statistics **Lecture Date: April 26, 2007**

More Functional Delta Method; Quantiles

Lecturer: Michael I. Jordan *Scribe: Chris Haulk*

1 Motivation

Last lecture we developed a functional delta method that using the notion of Gateaux derivative. With

$$\phi'_P(\delta_x - P) = \left(\frac{d}{dt}\phi((1-t)P + t\delta_x)\right)_{t=0} = IF_{\phi,P}(x)$$

we write

$$\phi(P_n) - \phi(P) = \frac{1}{n}\sum_i IF_{\phi,P}(X_i) + R_n$$

and hope to show that $E_P[IF_{\phi,P}(X)] = 0$, $\text{Var}_P[IF_{\phi,P}(X)] = \gamma^2$, and $\sqrt{n}R_n = o_p(1)$. Then the CLT gives

$$\sqrt{n}(\phi(P_n) - \phi(P)) \xrightarrow{d} \mathcal{N}(0, \gamma^2).$$

Showing that $E_P[IF_{\phi,P}(X)] = 0$ should not be too hard, and calculating a variance at some point probably cannot be avoided if we want to show asymptotic normality of $\sqrt{n}(\phi(P_n) - \phi(P))$. However, showing that $\sqrt{n}R_n = o_p(1)$ may be difficult, depending on ϕ and P. For a delta method that avoids this last step, we will modify our notion of derivative.

2 Delta Method via Hadamard Differentiability

Let $\mathbb{D}$ and $\mathbb{E}$ be normed linear spaces and suppose $\phi : \mathbb{D}_\phi \to \mathbb{E}$ where $\mathbb{D}_\phi \subset \mathbb{D}$. We say that ϕ is *Hadamard differentiable* at θ if there is a continuous, linear map $\phi'_\theta : \mathbb{D} \to \mathbb{E}$ such that

$$\left\|\frac{\phi(\theta + th_t) - \phi(\theta)}{t} - \phi'_\theta(h)\right\|_{\mathbb{E}} \to 0 \quad \text{as } t \downarrow 0$$

for every sequence $h_t \to h$ such that $\theta + th_t \in \mathbb{D}_\phi$ for all sufficiently small t. If it is possible to define ϕ'_θ only on a subset $\mathbb{D}_0 \subset \mathbb{D}$ and the sequences h_t above are restricted to have limits h in $\mathbb{D}_0$, then ϕ is said to be Hadamard differentiable *tangentially* to $\mathbb{D}_0$.

Theorem 1 (Delta Method). *(van der Vaart, 1998, 20.8) Let $\mathbb{D}$ and $\mathbb{E}$ be normed linear spaces. Let $\phi : \mathbb{D}_\phi \subset \mathbb{D} \to \mathbb{E}$ be Hadamard differentiable at θ tangentially to $\mathbb{D}_0$. Let $T_n : \Omega_n \to \mathbb{D}_\phi$ be maps such that $r_n(T_n - \theta) \xrightarrow{d} T$ for some sequence of numbers $r_n \to \infty$ and a random element T that takes values in $\mathbb{D}_0$. Then $r_n(\phi(T_n) - \phi(\theta)) \xrightarrow{d} \phi'_\theta(T)$.*

Proof. Define $g_n(h) = r_n(\phi(\theta + h/r_n) - \phi(\theta))$ for $h \in \{h : \theta + h/r_n \in \mathbb{D}_\phi\}$. By Hadamard differentiability, $g_{n'}(h_{n'}) \to \phi'_\theta(h)$ for every subsequence $h_{n'} \to h \in \mathbb{D}_0$. Therefore $g_n(r_n(T_n - \theta)) \xrightarrow{d} \phi'_\theta(T)$ by the extended continuous mapping theorem 18.11. □

3 Applications and Examples

Last lecture we used the Gateaux functional delta method to prove asymptotic normality of the Mann-Whitney test statistic. We will prove this fact again using the Hadamard version of the functional delta method.

Lemma 2. *(van der Vaart, 1998, Lemma 20.10) Let $\phi : [0,1] \to \mathbf{R}$ be twice continuously differentiable. Then the function $(F_1, F_2) \mapsto \int \phi(F_1) dF_2$ is Hadamard-differentiable at every pair of functions (F_1, F_2) such that $F_i \in D[-\infty, \infty]$ and F_i has bounded variation. The derivative is*

$$(h_1, h_2) \mapsto h_2 \phi \circ F_1 |_{-\infty}^{\infty} - \int h_{2-} d\phi \circ F_1 + \int \phi'(F_1) h_1 dF_2.$$

Here, h_- denotes the left-continuous version of h.

Proof. See text. □

Now suppose at time ν we observe two independent random samples $X_1, ..., X_{m_\nu}$, $Y_1, ..., Y_{n_\nu}$ from distributions F and G, respectively. Let $N_\nu = m_\nu + n_\nu$ and suppose $m/N \to \lambda \in (0,1)$ as $\nu \to \infty$. By Donsker's theorem and Slutsky's lemma,

$$\sqrt{N}(F_m - F, G_m - G) \xrightarrow{d} \left(\frac{\mathbb{G}_F}{\sqrt{\lambda}}, \frac{\mathbb{G}_G}{\sqrt{1-\lambda}} \right)$$

for independent Brownian bridges $\mathbb{G}_F$ and $\mathbb{G}_G$. Let $\phi(x) = x$ and apply Lemma 20.10 together with the functional delta method to see

$$\sqrt{N} \left(\int F_m dG_n - \int F dG \right) \xrightarrow{d} - \int \frac{\mathbb{G}_{G-}}{\sqrt{1-\lambda}} dF + \int \frac{\mathbb{G}_F}{\sqrt{\lambda}} dG.$$

That the limit distribution is Gaussian follows from a generalization of a well-known result for finite dimensional processes, namely that continuous linear transformations of Gaussian processes are Gaussian. Alternatively, note that Thm. 20.8 implies that the limit variable is the limit in distribution of

$$- \int \sqrt{N}(G_n - G)_- dF + \int \sqrt{N}(F_m - F) dG,$$

rewrite the expression above as a difference of scaled, centered sums, and apply the usual CLT.

4 Quantiles

The quantile function $F^{-1} : (0,1) \to \mathbb{R}$ of a cumulative distribution function F is

$$F^{-1}(p) = \inf\{x : F(x) \geq p\}.$$

The quantile function has some nice properties:

Lemma 3. *(van der Vaart, 1998, Lemma 21.1) For $0 < p < 1$ and $x \in \mathbb{R}$,*

- $F^{-1}(p) \leq x$ *iff* $p \leq F(x)$
- $F \circ F^{-1}(p) \geq p$
- $F^{-1} \circ F(x) \leq x$

- $F_- \circ F^{-1}(p) \leq p$ *where* F_- *denotes the left continuous version of* F
- $F^{-1} \circ F \circ F^{-1} = F^{-1}$
- $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$

Proof. Chase definitions, or see the text. □

In the next lecture we will see that p^{th} quantiles are asymptotically normal whenever F is differentiable with positive derivative at $F^{-1}(p)$, that is,

$$\sqrt{n}(F_n^{-1}(p) - F^{-1}(p)) \xrightarrow{d} N\left(0, \frac{p(1-p)}{f^2(F^{-1}(p))}\right).$$

For now we will merely calculate the influence function of $\phi_p(F) = F^{-1}(p)$. Assume that $F \circ F^{-1}(p) = p$. Let $F_t = (1-t)F + t\delta_x$. By the definition of F_t, the equality $p = F_t \circ F_t^{-1}(p)$ can be rewritten as

$$p = (1-t)F(F_t^{-1}(p)) + t\delta_x(F_t^{-1}(p)).$$

Differentiating both sides with respect to t, we get

$$0 = -F(F_t^{-1}(p)) + (1-t)f(F_t^{-1}(p))\frac{d}{dt}F_t^{-1}(p) + \delta_x F_t^{-1}(p) + t\delta_x \frac{d}{dt}F_t^{-1}(p)$$

and setting $t = 0$, we can solve for the influence function $(\frac{d}{dt}F_t^{-1}(p))_{t=0}$,

$$IF_{\phi,F}(x) = -\frac{1(F^{-1}(p) \geq x) - p}{f(F^{-1}(p))}.$$

References

van der Vaart, A. W. (1998). *Asymptotic Statistics.* Cambridge University Press, Cambridge.