WOP Summation Proof

Problem 1: Prove the following using a proof by contradiction that utilizes the Well-Ordering Principle.

Theorem: $\forall n. \in \mathbb{N} \setminus \{0, 1, 2, 3\}$

$$\sum_{i=4}^{n} 4i = 2n^2 + 2n - 24$$

We proceed by contradiction. That is, suppose that there exists some natural number greater than 3 such that the theorem $\sum_{i=4}^{n} 4i = 2n^2 + 2n - 24$, which we hereby will refer to as $P(n)$, evaluates to False. Define the set C to contain all such counterexamples. Since our assumption is that there is at least one natural number greater than 3 for which this theorem does not hold, we know that C is a non-empty set.

Since C is a non-empty subset of the natural numbers, it is a well-ordered set. This means that the well-ordering principle applies, and there must be a smallest element of C. Define natural number x to be this minimum element of C.

If natural number x is the smallest element of C, meaning the smallest counter-example where the equation does not hold, then the next smallest element is x−1, and $P(x-1)$ must be true.

$$P(x-1): \quad \sum_{i=4}^{x-1} 4i = 2(x-1)^2 + 2(x-1) - 24 = 2(x^2 - 2x + 1) + 2x - 2 - 24 = 2x^2 - 2x - 24$$

We can also observe that when $n = x$, we have the following inequality:

$$P(x): \quad \sum_{i=4}^{x} 4i \neq 2x^2 + 2x - 24$$

Next, by algebra, we consider that adding $4(x)$ to both sides of an equation preserves equality, so the following equation must hold:

$$\hookrightarrow P(x-1): \quad \sum_{i=4}^{x-1} 4i = 2x^2-2x-24$$

$$\begin{array}{rcl} +(4x) & & +(4x) \\ \hline \sum_{i=4}^{x} 4i & = & 2x^2-2x-24+4x = 2x^2+2x-24 \end{array}$$

However, therein lies the contradiction, because the equation that we derived cannot be true at the same time as the fact $\neg P(x)$. That is, the above equation in conjunction with the statement that x is the smallest counter example can be represented as follows:

$$\Big(\underbrace{\sum_{i=4}^{x} 4i = 2x^2+2x-24}_{\text{derived from } P(x-1)}\Big) \land \Big(\underbrace{\sum_{i=4}^{x} 4i \neq 2x^2+2x-24}_{\neg P(x)}\Big) \equiv \perp$$

So far, this means that $\forall n \in \mathbb{N} \setminus \{0,1,2,3\}.\ P(n) \leftrightarrow P(n-1)$

However, our proof currently does not take into account that perhaps there is no $x-1$ that is smaller than x and also a natural number that is greater than 3. There is exactly one case where this could occur:

When x is the smallest natural number that is greater than 3, that is $x=4$.

However, we can directly evaluate $P(4)$.

LHS: $\sum_{i=4}^{4} 4i = 4(4) = 16$

RHS: $2(4)^2 + 2(4) - 24 = 32 + 8 - 24 = 16$

$\hookrightarrow 16 = 16$ ✓

We know that the smallest positive natural number that is greater than 3 is 4, and $P(4)$ is true, since the summation evaluated when $n=4$ is equal to 16, and the explicit formula is also equal to 16.

Therefore, we have derived a contradiction: we have proven that across all the positive integers that if $P(n)$ holds, then $P(n-1)$ must also be true. In logic, that is:

$$\forall n \in \mathbb{N} \setminus \{0, 1, 2, 3\}. \, (P(n) \longleftrightarrow P(n-1)) \wedge P(4) \equiv \forall n \in \mathbb{N} \setminus \{0, 1, 2, 3\}. \, P(n)$$

$\blacksquare$