# MAT 127: Calculus C, Fall 2010

## Notes on the Ratio Test: for Sequences and Series

## 0 Introduction

In the textbook, the *Ratio Test* is introduced in Section 8.4 as one of many convergence tests for *series*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots .$$

A convergence test for *sequences* $a_1, a_2, \ldots$ can be deduced from *RT for Series* using the *Divergence Test for Series* (box 6 or 7 on p570). However, this convergence test can also be obtained more directly. Just like *RT for Series*, *RT for Sequences* works well for some sequences and is useless for others. On the other hand, *RT for Series* is perhaps the most frequently used convergence test for series, while *RT for Sequences* is perhaps the least frequently used convergence test for series (but it does work very nicely in some cases!).

In order to use either of the ratio tests, look at the ratio of the absolute values of two consecutive terms, $|a_{n+1}|/|a_n|$; note that the higher-numbered term goes to the top of this fraction and the lower-numbered term goes to the bottom. For example, for the sequence $a_n = (-1)^n 7^n/n!$, this ratio is

$$\frac{|a_{n+1}|}{|a_n|} = \frac{7^{n+1}/(n+1)!}{7^n/n!} = \frac{7^{n+1}}{7^n} \cdot \frac{n!}{(n+1)!} = \frac{7^n \cdot 7}{7^n} \cdot \frac{n!}{n! \cdot (n+1)} = \frac{7}{n+1} .$$

This produces a new sequence $a_2/a_1, a_3/a_2, a_4/a_3, \ldots$, which consists of non-negative terms.

## 1 Ratio Test for Sequences

First, suppose we want to determine whether the *sequence* $a_1, a_2, \ldots$ has a limit. We could instead look at the sequence of the absolute values of the ratios of consecutive terms $|a_{n+1}|/|a_n|$. If the latter sequence has a limit $L$ (which must necessarily be non-negative) and

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} < 1, \qquad \textbf{then} \qquad \lim_{n\longrightarrow\infty} a_n = 0. \tag{1}$$

On the other hand, if

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \quad \text{or} \quad \frac{|a_{n+1}|}{|a_n|} \longrightarrow \infty, \qquad \textbf{then} \qquad |a_n| \longrightarrow \infty, \tag{2}$$

and so the sequence $a_n$ diverges (or possibly "converges" to infinity). Finally, if

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = 1, \qquad \textbf{then} \qquad \text{this test says nothing.} \tag{3}$$

In this last case, you'll need to find some other way to determine if the sequence $a_1, a_2, \ldots$ converges.

For example, for the sequence $a_n = (-1)^n 7^n/n!$,

$$\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\longrightarrow\infty} \frac{7}{n+1} = \frac{7}{\infty+1} = 0.$$

Since $0 < 1$, this sequence converges to 0. On the other hand, for the sequence $a_n = 2^n/n$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}/(n+1)}{2^n/n} = \frac{2^{n+1}}{2^n} \cdot \frac{n}{n+1} = \frac{2^n \cdot 2}{2^n} \cdot \frac{1}{n/n + 1/n} = 2\frac{1}{1+1/n} \longrightarrow 2\frac{1}{1+1/\infty} = 2\frac{1}{1+0} = 2.$$

Since $2 > 1$, this sequence diverges (actually "converges" to $\infty$).

For the sequence $1, 1, 1, \ldots$, the limit of the ratios of the absolute values of consecutive terms is 1 (all of these ratios are 1) and this sequence converges 1. For the sequence $-1, 1, -1, 1 \ldots$, the limit of the ratios of the absolute values of consecutive terms is also 1 (all of these ratios are again 1), but this sequence diverges (it keeps on jumping between 1 and -1). This shows that *RT for Sequences* is useless in the case

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = 1.$$

Even a sequence $a_1, a_2, a_3, \ldots$ with $L = 1$ *and* $a_n > 0$ need not converge. For example, let

$$a_1 = 2^{\frac{1}{1}}, \quad a_2 = 2^{\frac{1}{1}+\frac{1}{2}}, \quad a_3 = 2^{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}}, \quad \ldots. \tag{4}$$

Then,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}+\frac{1}{n+1}}}{2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}} = \frac{2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}} \cdot 2^{\frac{1}{n+1}}}{2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}} = 2^{\frac{1}{n+1}} \longrightarrow 2^{\frac{1}{\infty+1}} = 2^0 = 1.$$

However,

$$2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}} \longrightarrow 2^{\infty} = \infty,$$

because the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$$

diverges by the *p-Series Test* on p578 (this is $p = 1$; see also Example 7 on p569). Thus, the sequence of positive terms $a_1, a_2, \ldots$ described in (4) diverges (it actually "converges" to $\infty$), even though $L = 1$ in this case.

*Why is (1) true?* The assumption in (1) is that the ratios $|a_{n+1}|/|a_n|$ get very close to $L$ as $n$ increases. Since $L < 1$ in this case, this means that $|a_{n+1}|/|a_n| < (L+1)/2$ if $n$ is very large, so that

$$|a_{n+1}| < \frac{L+1}{2}|a_n|$$

for all $n$ larger than some $N$. Thus,

$$|a_{N+n}| < \frac{L+1}{2}|a_{N+n-1}| < \frac{L+1}{2} \cdot \frac{L+1}{2}|a_{N+n-2}| \ldots < \underbrace{\frac{L+1}{2} \cdot \ldots \cdot \frac{L+1}{2}}_{n}|a_N| = \left(\frac{L+1}{2}\right)^n |a_N|.$$

Thus, the sequence $|a_N|, |a_{N+1}|, \dots$ is squeezed between the sequence $0, 0, \dots$ and the geometric sequence

$$|a_N|,\ \frac{L+1}{2}|a_N|,\ \left(\frac{L+1}{2}\right)^2|a_N|,\ \dots$$

This geometric sequence converges to 0 because $|(L+1)/2| < 1$ in this case (see box 7 on p560). Thus, the sequence $|a_N|, |a_{N+1}|, \dots$ also converges to 0 by the *Squeeze Theorem for Sequences* on p557 and so does the sequence $a_N, a_{N+1}, \dots$ by Theorem 4 on p557. Since the convergence of a sequence has nothing to do with how it begins, it follows that the original sequence $a_1, a_2, \dots$ also converges.

*Why is (2) true?* The assumption in (2) is that the ratios $|a_{n+1}|/|a_n|$ become larger than some number $r > 1$ as $n$ increases (in the first case, we can take $r = (L+1)/2$; in the second case, $r$ can be taken to be *any* number larger than 1). Thus, $|a_{n+1}| > r|a_n|$ for all $n$ larger than some $N$ and so

$$|a_{N+n}| > r|a_{N+n-1}| > r \cdot r|a_{N+n-2}| \dots > \underbrace{r \cdot \ldots \cdot r}_{n}|a_N| = r^n|a_N|.$$

So, the terms in the sequence $|a_N|, |a_{N+1}|, \dots$ are larger than the terms in the geometric sequence

$$|a_N|,\ r|a_N|,\ r^2|a_N|,\ \dots$$

This geometric sequence diverges (actually "converges" to $\infty$) because $r > 1$ (see box 7 on p560). Thus, the sequence $|a_N|, |a_{N+1}|, \dots$ also diverges (also "converges" to $\infty$), since its terms are even larger. Since the convergence of a sequence has nothing to do with how it begins, it follows that the original sequence $a_1, a_2, \dots$ also diverges.

Because of (1), (2), and (3), *RT for Sequences* can detect convergence of sequences $a_1, a_2, \dots$ with limit 0 only (and even of only some of these). So it can rarely be used to detect convergent sequences, but whenever it is applicable, *RT for Sequences* determines the limit of convergent sequences immediately. *RT for Sequences* has a good chance of working for sequences that involve factorials and powers $n$ (e.g. $n!$, $3^n$, $n^n$), but has *no* chance of working for sequences that involve only powers of $n$ (e.g. $n^3$).

## 2 Ratio Test for Series

*RT for Series* works similarly to *RT for Sequences* to determine whether certain series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

converge. As before, we look at the sequence of the absolute values of the ratios of consecutive terms $|a_{n+1}|/|a_n|$ (still sequence, not series!). If this sequence has a limit $L$ (which must necessarily be non-negative) and

$$L \equiv \lim_{n \longrightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1, \qquad \textbf{then} \qquad \sum a_n \ \text{ converges.} \tag{5}$$

On the other hand, if

$$L \equiv \lim_{n \longrightarrow \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \quad \text{or} \quad \frac{|a_{n+1}|}{|a_n|} \longrightarrow \infty, \qquad \textbf{then} \qquad \sum a_n \ \text{ diverges.} \tag{6}$$