11-16-2009

Fractions and the Set of Rational Numbers

We introduced the whole numbers as a system for keeping track of the number of objects in a set. Later on, we discovered that negative numbers could help us to, for example, track debits and credits in our bank account, and so we introduced the *integer* number system. Our task in this section is to describe *fractions* which help us to count parts of whole objects. We then present the *rational* number system which is the set of all fractions.

Definition. A **fraction** is an ordered pair of integers a and b with $b \neq 0$ which we write in the form $\frac{a}{b}$ or a/b. The a is called the **numerator** and the b is called the **denominator**.

Models for Fractions. A good model for fractions must do the following:

1. Specify the unit or "whole".
2. Describe how many equal parts the unit has been subdivided into. This gives the denominator.
3. Describe how many parts of the whole are present. This gives the numerator.

Some possible models are:

1. Colored regions. Choose a shape. Divide the shape into *equal* parts. The number of parts is the denominator of your fraction. Color the number of parts for the numerator. See page 346 for some example pictures.

2. Set model. Draw your universe set U so that the number of objects in the universe is the number in your denominator. Collect all the objects that you are trying to count in a set A. Then the corresponding fraction is

$$\frac{n(A)}{n(U)}$$

see the diagram on page 347.

3. Fraction strips. Give each student a strip of paper. To represent the fraction $\frac{a}{b}$, have the students use a pencil to subdivide their strips into b equal rectangles. Then have them color a of the rectangles.

4. Number lines. To represent the fraction $\frac{a}{b}$, take the standard integer number line and divide each of the intervals between the integers into b equal subintervals. Then, beginning at zero, count out a subintervals on the number line. See page 348.

Basic Properties of Fractions.

Definition. Two fractions are called **equivalent** if they represent the same quantity.

Example. Make fractions strips for the following fractions:

$$\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots$$

The n$^{\text{th}}$ fraction of this sequence is $\frac{2 \cdot n}{3 \cdot n}$. Moreover by our fraction strips each of these fractions are really **equivalent**. That is,

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \ldots = \frac{2 \cdot n}{3 \cdot n}.$$

In general, this is the **fundamental property of fractions**:

Theorem. for $\frac{a}{b}$ a fraction and $n \neq 0$ an integer we have that

$$\frac{a}{b} = \frac{an}{bn}.$$

In words this says that if I multiply the numerator and the denominator of a fraction by any non-zero integer I get an equivalent fraction back.

Now notice that

$$\frac{35}{21} = \frac{5 \cdot 7}{3 \cdot 7} = \frac{5}{3}$$

by the fundamental property of fractions. Thus, to reduce a fraction we can factor the numerator and the denominator and then "cancel out" all the common factors.

Suppose I give you two fractions. How can you tell if they are equivalent? The rule is given by the following theorem.

Theorem. The fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$. That is,

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc.$$

So are $\frac{3}{12}$ and $\frac{2}{8}$ equivalent?

A good model for these ideas are the pizza diagrams. See page 350.

Definition. A fraction is in **simplest form** if a and b have no common divisor larger than 1 and b is positive.

Example. Write $\frac{360}{600}$ is simplest form.

An easy way to do this is to successively divide out the common factors of the numerator