# Math 216: Differential Equations

## Lab 2: Euler's Method and RC Circuits

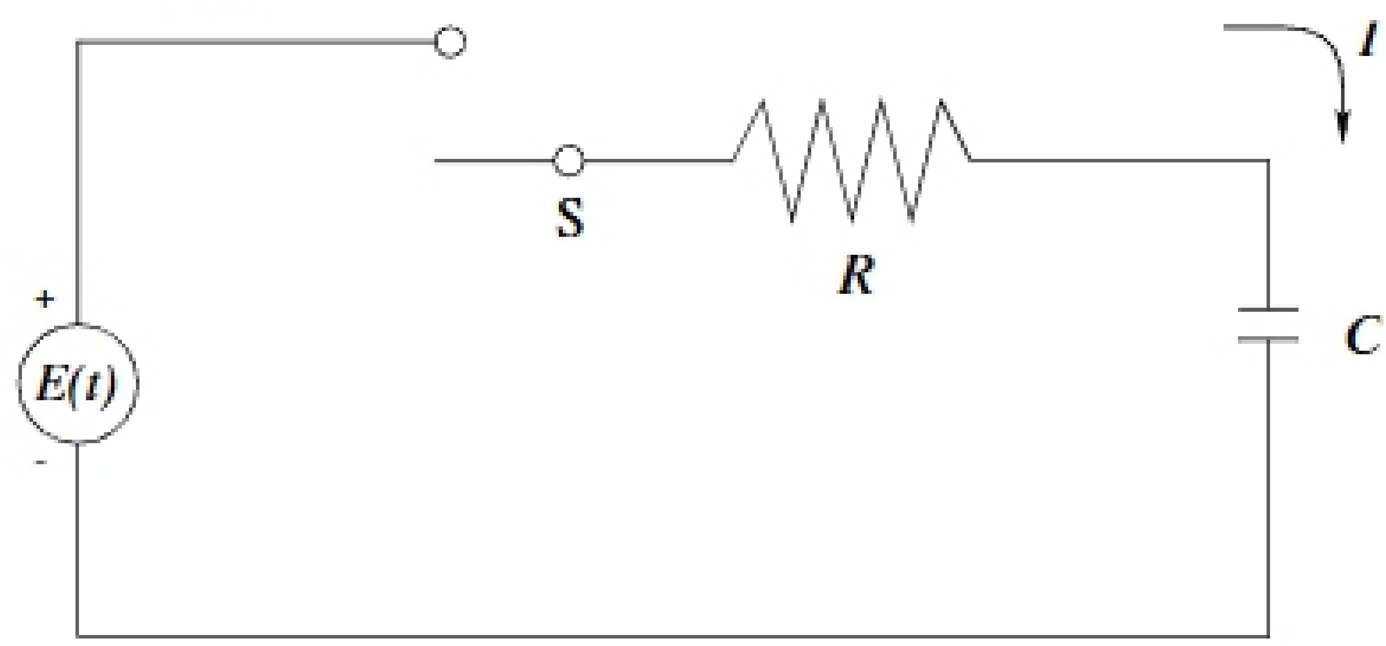
### Goals

In this lab you will implement Euler's method to approximate measurements of the charge on a capacitor in a basic RC circuit. You will learn how to write `.m` files for Matlab and how to program Euler's method; then you will investigate some of the limitations of Euler's method.

### Application: a basic RC circuit

The state of an electrical circuit consisting of resistors, capacitors, and an applied voltage can be described by differential equations.

Consider the following circuit




with resistance $R$ Ohms ($\Omega$), capacitance $C$ Farads (F), and applied voltage $E(t)$ Volts (V). The charge on the top plate of the capacitor at time $t$ is $Q(t)$ Coulombs (C), and the current through the resistor is $I(t)$ Amperes (A). The resistor has a voltage drop of $RI$, and the capacitor has a voltage drop of $Q/C$. When switch $S$ is closed at time $t = 0$, the sum of the voltage across the resistor and the capacitor must equal the applied voltage. This gives us the equation

$$RI + \frac{1}{C}Q = E(t)\,.$$

The current in the circuit is the rate of change of the amount of charge on the capacitor. So using the relationship $dQ/dt = I$, this becomes a first order differential equation for $Q(t)$:

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)\,.$$

The initial condition for this equation is $Q_0 = Q(0)$, the initial amount of charge on the capacitor. ($Q_0$ could be set by imposing a voltage $V_0 = Q_0/C$ across the capacitor before inserting it into the circuit.) In this lab, we will use Euler's method to numerically solve this differential equation for two different applied voltages: a constant voltage $E_0$, and then an AC voltage $E(t) = 117\sin(120\pi t)$ which corresponds to the voltage out of a standard wall socket in the United States.

## Prelab assignment

Before arriving in lab, answer the following questions. You will need your answers in lab to work the problems, and your recitation instructor may check that you have brought them. These problems are to be handed in as part of your lab report.

1. (a) We first consider the case where $E(t) = E_0$, a constant applied voltage. Verify that the function

$$Q_1(t) = E_0 C\left(1 - e^{-t/(RC)}\right) \tag{1}$$

is a solution to the initial value problem with $E(t) = E_0$,

$$\frac{dQ_1}{dt} = -\frac{1}{RC}Q_1 + \frac{1}{R}E_0\,, \quad Q_1(0) = 0\,, \tag{2}$$

where $R$, $C$, and $E_0$ are constants. That is, by plugging $t = 0$ into the formula (1) show that the initial condition is satisfied, and then by differentiating the formula (1) and comparing with the right-hand side of the differential equation show that $Q_1(t)$ satisfies the differential equation. (In other words, do not try to find the solution of the initial-value problem, but rather just check that the given function solves the problem.) Then use the exact solution formula (1) with $R = 18000\Omega$, $C = 0.0000125$F,and $E_0 = 117$V to complete the column labelled "Exact $y$" on Table 2 on the last page of the lab, for use in Lab problem 1.

   (b) Next consider the case where $E(t) = E_0\sin(120\pi t)$. Verify that the function

$$Q_2(t) = \frac{E_0}{R\left(\gamma^2 + (120\pi)^2\right)}\left[120\pi\left(e^{-\gamma t} - \cos(120\pi t)\right) + \gamma\sin(120\pi t)\right] \tag{3}$$

satisfies the initial-value problem with this $E(t)$,

$$\frac{dQ_2}{dt} = -\frac{1}{RC}Q_2 + \frac{1}{R}E_0\sin(120\pi t)\,, \quad Q_2(0) = 0\,, \tag{4}$$

where the constant $\gamma = 1/(RC)$ has units of Hz=sec$^{-1}$. What do the initial condition and forcing term $\frac{1}{R}E_0\sin(120\pi t)$ represent for the system at the time the switch is closed?

2. Suppose you implement Euler's method using Matlab, using step size $h$, and create a vector `t` of time steps from $t = 0$ to $t = 1$. Often we refer to the first entry as $t_0 = 0$, the next as $t_1$ and the final entry will be $t_N = 1$ where $Nh = 1$. Matlab does not enumerate these entries in the same way. The first element of a vector in Matlab is always `t(1)`. In this case, we will have `t(1)=0`, and `t(N+1)=1`. Find the Matlab indices `n` so that `t(n)=0`, `t(n)=.5`, `t(n)=.86`, and `t(n)=1` if you used

   (a) $N = 12$ (Note: you cannot get `t(n)=.86` in this case.)

   (b) $N = 120$ (Note: you cannot get `t(n)=.86` in this case.)

   (c) $N = 1200$

   (d) $N = 2400$

   Record these values of `n` in Table 1 below.

| $N$ | `n` so that `t(n)=0` | `n` so that `t(n)=.5` | `n` so that `t(n)=.86` | `n` so that `t(n)=1` |
|---|---|---|---|---|
| 12 | | | | |
| 120 | | | | |
| 1200 | | | | |
| 2400 | | | | |

Table 1: Matlab Indices for Time Vector

## In the lab

The primary manner in which we will work with Matlab is by writing simple programs that we can then run in Matlab. Each program is a small file with a `.m` extension. For this lab we will write programs that implement Euler's method, and will compare the results obtained for several step sizes $h$ with exact solutions (when these are available).

### Creating a (new) file

Launch Matlab. Use the buttons in the "Current Folder" box to the left of the Matlab window to make sure that you are working on the Desktop. *Note that we will work with files on the Desktop in this lab—but that you will need to be sure to copy them to your M+Box space or Google Drive cloud space (or Dropbox, or a thumb drive, etc.) to have them for future use after you finish in the lab today. Anything you do not move to one of these* ***will be lost***. Start editing a file by going to the File menu and selecting New and Script. A new window called untitled will open; give it a name by selecting File→Save As in this window; give your file the name `EULER.m` and save it. You have now created an (empty) file called `EULER.m`. Note that capitalization matters here.