Relativistic Dynamics: The Relations Among Energy, Momentum, and Velocity of Electrons and the Measurement of e/m

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This experiment is a study of the relations between energy, momentum and velocity of relativistic electrons. Using a spherical magnet generating a uniformly vertical magnetic field to accelerate electrons around a circular path and into a narrow slit, you will be able to predict relationships between properties of high velocity electrons. The experiment's goal is to compare your data with the models predicted by nonrelativistic and relativistic dynamics. Additionally you will be able to determine the value of the electron charge to mass ratio, the electron mass, and subsequently the electron charge.

PREPARATORY QUESTIONS

Please visit the Relativistic Dynamics chapter on the 8.13x website at mitx.mit.edu to review the background material for this experiment. Answer all questions found in the chapter. Work out the solutions in your laboratory notebook; submit your answers on the web site.

UNITS IN THIS LAB GUIDE

This lab guide uses Gaussian units in which the force on a moving charge in a static field is

$$\vec{F} = q\vec{E} + \frac{q}{c}(\vec{v} \times \vec{B}).$$

The units are:
F — dyne
q — statcoulomb (esu)
B — gauss
E — statvolts cm^{-1}
v, c — cm sec^{-1}
Be aware that while magnetometers often display magnetic field values in gauss, voltmeters almost exclusively display voltage in volts, not statvolts.

Some useful constants:

$$e = 4.80298 \times 10^{-10} \text{ statcoulombs}$$
$$mc^2 = 511 \text{ keV} = 8.18727 \times 10^{-7} \text{ ergs}$$
$$1 \text{ volt} = 3.336 \times 10^{-3} \text{ statvolts}$$

SUGGESTED PROGRESS CHECK AT END OF 2^{nd} SESSION

With the spectrometer magnetic field set to 100 gauss, plot the electron count rate at the detector versus velocity selector voltage (converted to units of β). Also, plot at least three data points of various kinematic energies versus β.

I. INTRODUCTION

Between 1900 and 1910 the relation between the energy, momentum and velocity of charged particles moving at high speeds was a central problem of physics. The fundamental contradictions between Newtonian mechanics and the Maxwell theory of the electromagnetic field, revealed most dramatically in the failures of the Michelson-Morley experiment to detect absolute motion of the earth through the "aether", barred the way to a logically consistent understanding of the deflection of a charged particle by electric and magnetic fields when the particle is moving at a velocity approaching the velocity of electromagnetic waves. Various formulas were derived by Abraham, Lorentz, and Poincare. In 1901 Kaufmann, using the new vacuum techniques pioneered by Thomson, determined the "apparent mass of the electron" by measuring the deflections of the recently discovered β rays from radioactive substances. There was considerable confusion as to whether the experimental results confirmed or contradicted one or another of the formulas. Then Albert Einstein, at the time an obscure 25-year-old examiner in the Swiss patent office, provided a clear and compelling theory of "the electrodynamics of moving bodies" which came to be called the special theory of relativity. Among its remarkable predictions was the slowing of clocks moving at high speed (demonstrated in the Junior Lab experiment on muon decay), and the non-classical relations between momentum, energy and velocity that are demonstrated in the present experiment [1].

In nonrelativistic dynamics a particle acted on by a force $\vec{F}$ for a time dt over a displacement $d\vec{r}$ gains momentum $d\vec{p} = \vec{F}dt$ and kinetic energy $dK = \vec{F} \cdot d\vec{r}$.

According to Newtonian dynamics the kinetic energy K, momentum $\vec{p}$, and velocity $\vec{v}$ of a particle are related by the equations

$$\vec{p} = m\vec{v} \tag{1}$$

and

$$K = p^2/2m, \tag{2}$$

where $p^2 = \vec{p} \cdot \vec{p}$ and m is the inertial mass of the particle. In nonrelativistic mechanics there is no limit on the magnitude of $\vec{v}$.

According to relativistic dynamics (see [2], or other text on special relativity), these quantities are related by the equations

$$\vec{p} = m\gamma\vec{v} \tag{3}$$

and

$$E^2 = p^2c^2 + m^2c^4, \tag{4}$$

where $E = K + mc^2 = \gamma mc^2$ is the "total relativistic energy." The quantity m is a relativistic invariant and identical with the nonrelativistic inertial mass, and γ is the Lorentz factor defined by the equation

$$\gamma = (1 - \beta^2)^{-1/2}, \tag{5}$$

where $\beta = |\vec{v}|/c$. Solving equation (4) for the kinetic energy one obtains

$$K = mc^2\left(\left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2} - 1\right). \tag{6}$$

In the limit of high velocities where $p \gg mc$, equation (6) approaches

$$K = pc, \tag{7}$$

which is the exact relation between the energy and momentum of massless particles such as photons and neutrinos. For $p < mc$ equation (6), expanded by the binomial theorem, becomes

$$K = \frac{p^2}{2m}\left[1 - \frac{1}{4}\left(\frac{p}{mc}\right)^2 + \cdots\right]. \tag{8}$$

In the limit of low velocities ($|\vec{v}| \ll c$) where $p \ll mc$, Equation (8) reduces to the nonrelativistic relation expressed by Equation (2).

The electromagnetic force on a charged particle is

$$\vec{F} = q\left[\vec{E} + \left(\frac{\vec{v}}{c}\right) \times \vec{B}\right], \tag{9}$$

where q is the invariant charge, $\vec{E}$ and $\vec{B}$ are the electric and magnetic field intensities, respectively, and c is the invariant speed of light. This force law is valid in both nonrelativistic and relativistic dynamics. In this experiment you will measure the effects of electromagnetic force on the motion of electrons with velocities up to about $0.8c$, and measure directly the relations between $\vec{v}$, $\vec{p}$, and K.

Some useful references related to beta-decay, beta-spectroscopy and this experiment in particular are given in [3–8].

II. EXPERIMENTAL SETUP

The apparatus for the present experiment, shown schematically in Figure 1, is contained in a vacuum chamber inside a spherical magnet that maintains a uniform field in the vertical direction. Inside the vacuum chamber are

1. a source of energetic electrons (a minute quantity of ^{90}Sr which emits electrons with a spectrum of energies up to 0.546 MeV yielding a decay product ^{90}Y which emits electrons with a spectrum of energies up to 2.27 MeV),

2. baffles that reduce background counts due to scattered electrons that bounce around the vacuum chamber,

3. a narrow slit in the baffles at the 90-degree position around a circular path from the source,

4. a narrow slit that defines the radius of curvature of the electrons that enter the velocity selector, and

5. a solid state PIN diode detector.

The velocity selector has a length of 10 cm and a plate separation distance of 0.180 ± 0.003 cm.

An electron, emitted with a momentum in a narrow range of magnitude and direction, traverses a helical path of fixed radius in the magnetic field and enters the gap between the velocity-selector plates. The distance from the source to the velocity selector, which is also the diameter of the electron's path, is 40.6 ± 0.4 cm. If the voltage V between the plates is adjusted so that the electric force cancels the magnetic force, then the electron passes through the "velocity selector" in a nearly straight trajectory and strikes the PIN diode. In the diode, a number of silicon valence electrons proportional to the energy deposited by the electron are promoted to the conduction band of the semiconductor and collected as a pulse of charge. The latter is converted by a low-noise preamplifier and precision amplifier into a voltage pulse with an amplitude proportional to the deposited energy. This pulse amplitude is then measured and counted by a multichannel pulse height analyzer (MCA).

For each of several settings of the magnet current one measures

1. the magnetic field,

2. the voltage across the selector plates that yields the maximum counting rate of pulses in the narrow range of pulse height channels of the MCA corresponding to the energy range of the detected electrons, and

3. the median channel of the pulse height distribution.

Given these data, the dimensions of the apparatus, and the energy calibration of the PIN detector, one can determine the momentum, energy, and velocity of electrons and estimate the values of e/m, m, and c.

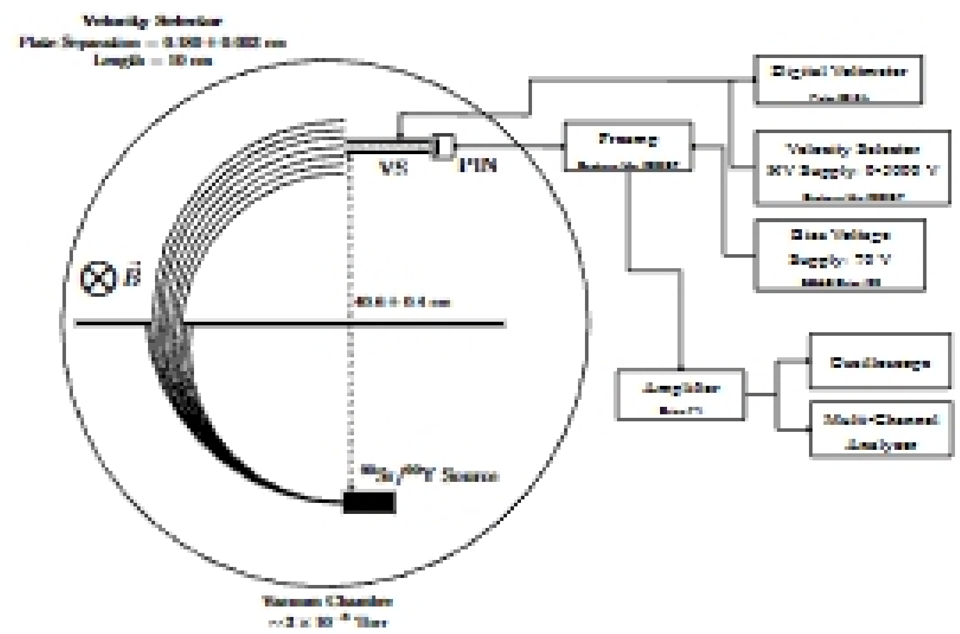

FIG. 1. Schematic diagram of the electron trajectory in the apparatus, the particle spectrometer and associated circuitry. The velocity selector is labeled VS, and the diode detector PIN.

II.1. Theory of the Apparatus

In the region between the source and the velocity selector, where only a magnetic field exists, the motion is described by the equation

$$\frac{e|\vec{v}|B}{c} = \left|\frac{d\vec{p}}{dt}\right| = \omega|\vec{p}| = \frac{|\vec{v}|p}{\rho}, \tag{10}$$

so

$$B = \left(\frac{c}{e\rho}\right) p, \tag{11}$$

where ρ is the radius of curvature of the particle trajectory under the influence of the magnetic force. Placement of the source, the collimator, and the aperture of the velocity selector on a circle of radius ρ allows only particles with a momentum in a narrow range around $Be\rho/c$ to enter the velocity selector. In the region between the velocity selector plates, $\vec{E}$, $\vec{B}$, and $\vec{v}$ are perpendicular to each other, so one can write the following relation for particles that experience zero deflecting force and go exactly parallel to the plates:

$$eE - \frac{evB}{c} = 0. \tag{12}$$

The voltage between the velocity selector plates is known from Faraday's law,

$$V = \oint \vec{E} \cdot d\vec{\ell}, \tag{13}$$

and therefore $E = V/d$. Hence,

$$\beta = |\vec{v}|/c = E/B. \tag{14}$$

Thus, for any combination of E and B such that $E < B$, the velocity selector transmits particles with velocities near E/B in a narrow range of magnitudes whose width depends on the geometry of the gap between the plates.

A plot of measured values of B against the ratio E/B reveals the relation between momentum and velocity. According to the nonrelativistic Equation (1), the plot would be fit by a straight line with a slope of $(mc^2)/(e\rho)$. Deviation from a straight line as $E/B \to 1$ indicates the failure of the nonrelativistic relation between momentum and velocity as the velocity approaches c. According to the relativity Equation (3), a plot of B against $(E/B)[1-(E/B)^2]^{-1/2}$ should be fit by a straight line with a slope of $(mc^2)/(e\rho)$. From the slope and knowledge of the values of c and ρ one can estimate the invariant quantity e/m.

In the experiment it is a good idea to set the magnetic field and then determine the voltage between the selector plates that gives the highest rate of counts of electrons that traverse the circular path and pass between the velocity selector plates to strike the PIN diode detector.

Note that measurements of E and B alone yield a determination of e/m but neither e nor m separately. This is characteristic of all experiments involving only electromagnetic forces. Why is this so? Consider the analogy to the problem of determining the ratio of gravitational to inertial mass of a body moving under the influence of gravity. (Incidentally, the 'Shot Noise' experiment in Junior Lab yields the measurement of e.)

The PIN diode detector combines the virtues of an ultra-thin entrance window and surface dead layer with a total sensitive thickness sufficient to stop electrons with several hundred keV of kinetic energy. Thus, with appropriate calibration, the PIN diode provides a measure of the kinetic energy of the detected electrons. Plots of the kinetic energy against E/B or against $[1-(E/B)^2]^{-1/2}$ reveal the relation between energy and velocity; the slope of the latter plot yields a value of m (or more conveniently mc^2 expressed in units of keV in terms of which the energies of the calibration X and gamma-ray photons are expressed). A plot of energy against B reveals the deviation of the energy-momentum relation from the nonrelativistic quadratic form $E = p^2/(2m)$ toward the linear form $E = pc$ valid for a particle moving with a velocity close to c.

II.2. Apparatus Details

The magnet consists of a stack of circular air-core coils enclosing a spherical volume and connected in series so as to produce a current distribution over the surface of the sphere which is approximately equal to the ideal smooth distribution required to produce a uniform field inside the sphere. It turns out that the required distribution of surface current density is

$$J(\theta, \phi) = J_0 \sin\theta, \tag{15}$$

where J_0 is related to the magnitude $B_i = |\vec{B_i}|$ of the uniform field inside by an equation that is left for the reader to derive (for hints, see Appendix A).