6.003: Signals and Systems

Frequency Response

October 20, 2009

Mid-term Examination #2

Wednesday, October 28, 7:30-9:30pm, Walker Memorial.

No recitations on the day of the exam.

Coverage: cumulative with more emphasis on recent material
lectures 1–12
homeworks 1–7

Homework 7 will include practice problems for mid-term 2.
However, it will not collected or graded. Solutions will be posted.

Closed book: 2 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.

Conflict? Contact freeman@mit.edu by Friday, October 23, 5pm.

Review

Last time, we saw how a linear, time-invariant (LTI) system can be characterized by its unit-sample/impulse response.

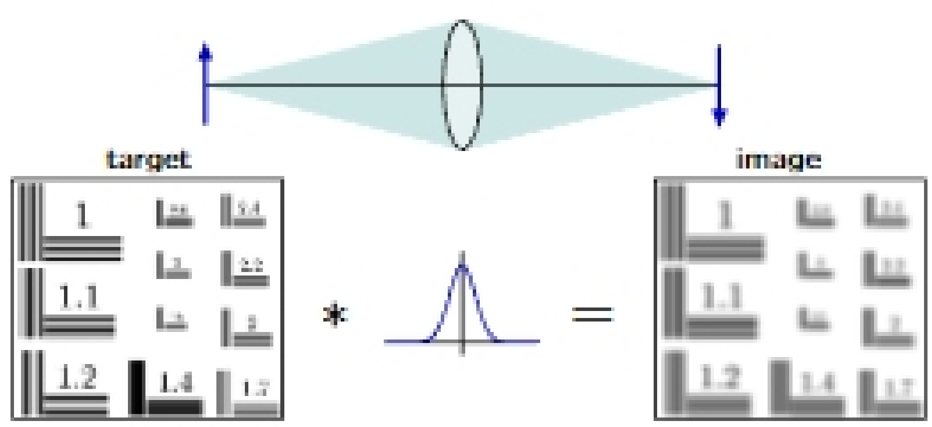
DT: $y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

CT: $y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Characterizing a system by its unit-sample/impulse response is especially insightful for some systems.

Microscope

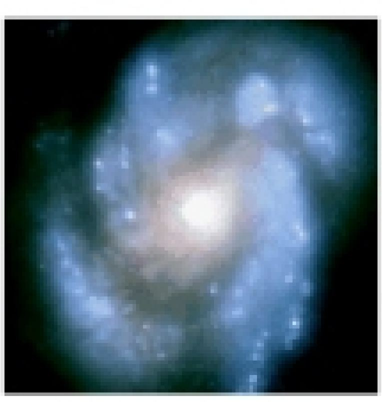
Blurring can be represented by convolving the image with the optical "point-spread-function" (3D impulse response).



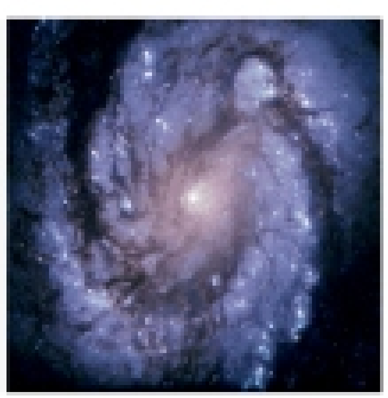
Blurring is inversely related to the diameter of the lens.

Hubble Space Telescope

Hubble images before and after upgrading the optics.



before



after

http://hubblesite.org

Frequency Response

Today we will investigate a different way to characterize a system: the **frequency response**.

Many systems are naturally described by their responses to sinusoids.

Example: audio systems

Check Yourself

How were frequencies modified in following music clips?

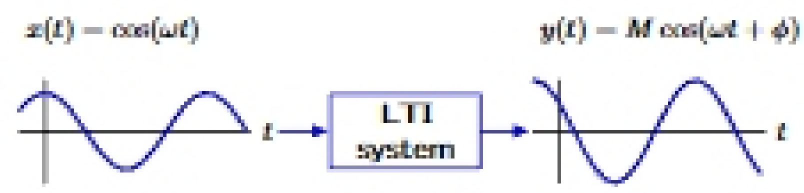
HF: high frequencies $\uparrow$: increased
LF: low frequencies $\downarrow$: decreased

	clip 1	clip 2
1.	HF$\uparrow$	HF$\downarrow$
2.	LF$\uparrow$	LF$\downarrow$
3.	HF$\uparrow$	LF$\downarrow$
4.	LF$\uparrow$	HF$\downarrow$
5.	none of the above	

Frequency Response Preview

If the input to a linear, time-invariant system is an eternal sinusoid, then the output is also an eternal sinusoid:

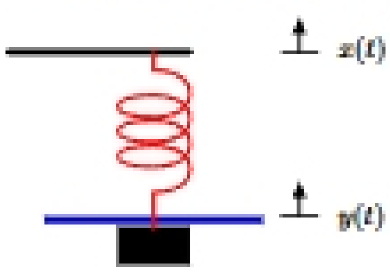
- same frequency
- possibly different amplitude, and
- possibly different phase angle.



The **frequency response** is a plot of the magnitude M and angle ϕ as a function of frequency ω.

Demonstration

Measure the frequency response of a mass, spring, dashpot system.



Frequency Response

Calculate the frequency response.

Methods

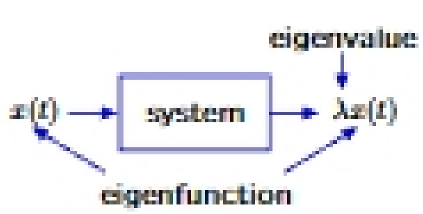
- solve differential equation
  $\rightarrow$ find particular solution for $x(t) = \cos\omega_0 t$
- find impulse response of system
  $\rightarrow$ convolve with $x(t) = \cos\omega_0 t$

New method

- use eigenfunctions and eigenvalues

Eigenfunctions

If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the scale multiplier as the eigenvalue.



Check Yourself: Eigenfunctions

Consider the system described by
$\dot{y}(t) + 2y(t) = x(t)$.
Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue.

1. e^{-t} for all time
2. e^{t} for all time
3. e^{jt} for all time
4. $\cos(t)$ for all time
5. $u(t)$ for all time

Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)e^{st}$$

Furthermore, the eigenvalue is $H(s)$!

Rational System Functions

If a system is represented by a linear differential equation with constant coefficients, then its system function is a ratio of polynomials in s.

Example:

$$\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 2\ddot{x}(t) + 7\dot{x}(t) + 8x(t)$$

Then

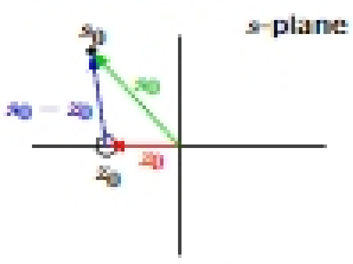
$$H(s) = \frac{2s^2 + 7s + 8}{s^2 + 3s + 4} = \frac{N(s)}{D(s)}$$

Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$



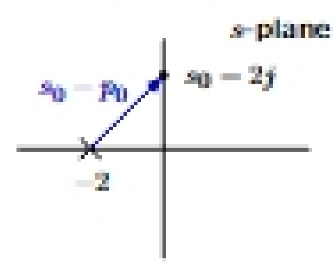
Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here z_0) to s_0, the point of interest in the s-plane.

Vector Diagrams

Example: Find the response of the system described by

$$H(s) = \frac{1}{s + 2}$$

to the input $x(t) = e^{2jt}$ (for all time).



The denominator of $H(s)|_{s=2j}$ is $2j + 2$, a vector with length $2\sqrt{2}$ and angle $\pi/4$. Therefore, the response of the system is

$$y(t) = H(2j)e^{2jt} = \frac{1}{2\sqrt{2}}e^{-\frac{j\pi}{4}}e^{2jt}.$$

Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

$$|H(s_0)| = |K|\frac{|(s_0 - z_0)||(s_0 - z_1)||(s_0 - z_2)|\cdots}{|(s_0 - p_0)||(s_0 - p_1)||(s_0 - p_2)|\cdots}$$

The angle is determined by the sum of the angles.

$$\angle H(s_0) = \angle K + \angle(s_0 - z_0) + \angle(s_0 - z_1) + \cdots - \angle(s_0 - p_0) - \angle(s_0 - p_1) - \cdots$$

Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos\omega_0 t$ (for all time). Then

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

and the response to a sum is the sum of the responses.

$$y(t) = \frac{1}{2}\left(H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}\right)$$