# Final Exam Review Sheet:

Time: 1:30-4:30 PM on Monday, May 11th, 2015
Place: Foellinger Auditorium
Sections: 7.1-7.5,7.7- 7.8, 8.1 - 8.3,10.1-10.4 , 11.1 - 11.11

- You have three hours for the exam.
- I recommend going over the list of materials mentioned in this sheet, then identifying which topics you are not comfortable with and working on the worksheets related to those topics. You should also solve all the sample exam questions.
- Take your time and go over the worksheets we did in class. In particular, you can work on the questions we didn't solve in class.
- Half of the exam is multiple choice so make sure you plan your time accordingly.
- Test taking strategy: Go through the multiple choice questions and solve only those you can solve without thinking for long. Then move on to the free response questions and solve them as careful as possible. Then, if you still have extra time go back to the multple choice questions you skipped.
- Don't forget to ask me if you have any questions.

# Review (Things you should definitely know):

The topics are split into three parts.

## Part I (Midterm 1):

1. **Substitution:** Although not officially part of the syllabus as it is in chapter 6 yet I think it is the cornerstone of everything we did in part I. Make sure you are comfortable with substitution techniques. This is very important. If you are not then you should most certainly work on it. You can also come and ask me if you think you need some help.

2. **Integration by Parts:** Remember the formula for the integration by parts:

$$\int u dv = uv - \int v du$$

   You can use it very often when we have a product of a polynomial and a trig (or exponential) function. You need to practice a lot so that you can identify $u$ and $v$ correctly, however, if you are not sure you can always try to use this formula:

   ***LIATE***

   which stands for *logarithmic (L), inverse trigonometric (I), algebraic (A), trigonometric (T), and exponential (E)* and the rule is:

   *Pick u in the order of LIATE.*

   However, remember there are always exceptions.

3. **Trig Integrals:** There are several standard trig integrals:

$$\bullet \int \sin^n x \cos^m x \, dx$$

If either $n$ or $m$ are odd, then you have to use substitution.
If $n$ or $m$ are both even, then we have to use the following formulas:

| | | |
|---|---|---|
| $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ | $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ | $\sin x \cos x = \frac{1}{2}\sin 2x$ |

$$\bullet \int \tan^m x \sec^n x \, dx$$

If $m$ is odd or $n$ is even, then use substitution.
If $m$ is even and $n$ is odd then there is no easy way. (But you should remember $\int \sec x \, dx$ and $\int \sec^3 x \, dx$).

There are trig integrals which are not of this form where determining the correct substitution needs some experience, see question 3 of worksheet 3.

4. **Trig Substitution:** Some integrals can be solved by substituting a trig function. These are the following cases:

| Expression | Substitution | $dx$ | Identity | Angle |
|---|---|---|---|---|
| $\sqrt{a^2-x^2}$ | $x = a\sin\theta$ | $dx = a\cos\theta d\theta$ | $a^2 - a^2\sin^2\theta = a^2\cos^2\theta$ | $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ |
| $\sqrt{a^2+x^2}$ | $x = a\tan\theta$ | $dx = a\sec^2\theta d\theta$ | $a^2 + a^2\tan^2\theta = a^2\sec^2\theta$ | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ |
| $\sqrt{x^2-a^2}$ | $x = a\sec\theta$ | $dx = a\sec\theta\tan\theta d\theta$ | $a^2\sec^2\theta - a^2 = a^2\tan^2\theta$ | $0 \leq \theta < \frac{\pi}{2}$ |

So, whenever you encounter these cases you can use trig substitution. However, you have to be careful as there are exceptions. Also, sometimes you have to first complete a square before you can use trig sub.

5. **Partial Fractions:** We introduce techniques to solve integrals of the form $\frac{P}{Q}$ such that $P$ and $Q$ are polynomials. First, we have to use polynomial division to make sure the degree of P is lower than Q. Then we split $Q = Q_1 Q_2 ... Q_n$ into its irreducible factors which can have three forms 1) $x-a$ or 2) $(x-a)^n$ (where $n$ positive) or 3) $ax^2+bx+c$ (where $b^2-4ac<0$). Depending on the form we now using following partical fraction equation:

- If $Q_i = x - a$ then we add $\frac{A}{x-a}$
- If $Q_i = (x-a)^n$ then we add $\frac{A_n}{(x-a)^n} + ... + \frac{A_1}{x-a}$
- If $Q_i = ax^2+bx+c$ where $b^2-4ac<0$ then we add $\frac{Cx+D}{ax^2+bx+c}$

For example:

$$\frac{P(x)}{(x-a)(x-b)^2(x^2+1)} = \frac{A}{x-a} + \frac{B_2}{(x-b)^2} + \frac{B_1}{(x-b)} + \frac{Cx+D}{x^2+1}$$

From there on we use ln and arctan to solve the integrals.

6. **Improper Integrals:** There are two general types:

Type 1:
- $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
- $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
- $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x) + \int_c^\infty f(x)\,dx$

Type 2:
- $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$ if f discontinuous at b
- $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$ if f discontinuous at a
- $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x) + \int_c^b f(x)\,dx$ If f discontinuous at c

There are some improper integrals which can't be solved so easily. For those cases we have some comparison theorems: Let $f$ and $g$ be continuous functions. Suppose that for all $x \geq a$, $0 \leq f(x) \leq g(x)$. Then:

- If $\displaystyle\int_a^\infty g(x)dx$ is *convergent*, then so is $\displaystyle\int_a^\infty f(x)dx$, and $\displaystyle\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$;
- if $\displaystyle\int_a^\infty f(x)dx$ is *divergent*, then so is $\displaystyle\int_a^\infty g(x)dx$.

7. **Strategies:** This list is not exhaustive and just covers general ideas. It is very important to practice a lot and gain as much experience as possible. There are many integrals which don't exactly fall into any of the mentioned categories. Also, many integrals require the use of several methods.

## Part II (Midterm 2):

8. **Approximating Integrals**: There are some integrals we cannot calculate easily and so we can use approximation formulas.

The Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}(f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n))$$

Simpson's Rule:

$$\int_a^b f(x)\,dx = \frac{b-a}{3n}(f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + 4f(x_{n-1}) + f(x_n))$$

where $x_i = a + i\Delta x$ and in the second case $n$ has to be even.