Q2 a) $p(x)y'' + G(x)y' + R(x)y = 0$

$p(x_0) = 0$ x_0 is singular point

$\Rightarrow y'' + p(x)y' + q(x)y = 0$

$p(x) = \frac{G(x)}{P(x)}$ $q(x) = \frac{R(x)}{P(x)}$

p_0 q_0

$(x-x_0)p(x)$ 和 $(x-x_0)^2 q(x)$ 在 $x \to x_0$ 时是 finite，则 x_0 是 regular singular point

b) $F(r) = r^2 + (p_0 - 1)r + q_0 = 0$

$p(x) = \frac{p_0}{x-x_0} + p_1 + p_2(x-x_0) + \dots$

$q(x) = \frac{q_0}{(x-x_0)^2} + \frac{q_1}{(x-x_0)} + q_2 + \dots$

解出 r_1, r_2 $y(x) = (x-x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$a_n(r) = -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} [(r+k)p_{n-k} + q_{n-k}] a_k$ $(n \geq 1)$

一定注意是否每个 n 能取到

c) 看定义域

例: $x^2 y'' + 2x^2 y' - 2y = 0$

$y'' + 2y' - \frac{2}{x^2} y = 0$

$F(r) = r^2 - r - 2 = (r-2)(r+1)$ $r_1 = -1$ $r_2 = 2$

$y_1(x) = x^{-1} \sum_{n=0}^{\infty} a_n(-1) x^n$ $y_2(x) = x^2 \sum_{n=0}^{\infty} a_n(2) x^n$

$p_1 = 2 \Rightarrow k = n-1$

$a_n(r) = -\frac{2(r+n-1)}{(r+n-2)(r+n+1)} a_{n-1}(r)$

for $r = -1$

$a_n = -\frac{2(n-2)}{(n-3)n} a_{n-1}$ $a_0 = 1$ $a_1 = -1$ $a_2 = 0$ $a_3 = 0 \dots$

$y_1(x) = x^{-1}(1-x) = \frac{1}{x} - 1$

for $r = 2$

$a_n = -\frac{2(n+1)}{n(n+3)} a_{n-1} = (-1)^n \frac{2^n (n+1)}{4 \cdot 5 \cdot 6 \cdots (n+3)}$

$y_2(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^n (n+1)}{(n+3)!} x^{n+2}$

$y(x) = c_1 y_1(x) + c_2 y_2(x)$ $c_1, c_2 \in R$

c) $(-\infty, 0)$ $y(x) = B)$

$y_1(x)$ is analytic at zero but $y_2(x)$ is not, the general solution on R

$y(x) = c y_1(x)$ $c \in R$

Q4 a) ① $\det(A - \lambda I)$ 求解 λ

对于三阶矩阵 $|\lambda I - A| = \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + k\lambda - |A|$

$k = (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}) - (a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32})$

② 对每个特征向量 $(A - \lambda I)\vec{v} = 0$ 求解特征向量 $\vec{v}$

③ $y(t) = e^{\lambda t}\vec{v}$ 复特征值

$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots$

b) $y' = Ay + b$ $y(t) = -A^{-1}b$

$\Leftrightarrow Ax = -b$

若 $y(t) = w_0 + t w_1$ $y_p(t)$

代入则可以

例: $A = \begin{bmatrix} 3 & -2 & -1 \\ -1 & 1 & 0 \\ 8 & -6 & -2 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\det(A - \lambda I) = \lambda(\lambda - 1)^2$ $\lambda_1 = 0$ $\lambda_2 = \lambda_3 = 1$

$A - 0I = \begin{bmatrix} 3 & -2 & -1 \\ -1 & 1 & 0 \\ 8 & -6 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\begin{bmatrix} 3a - 2b - c \\ -a + b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$A - I = \begin{bmatrix} 2 & -2 & -1 \\ -1 & 0 & 0 \\ 8 & -6 & -3 \end{bmatrix} \to \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$(A - I)\vec{v_3} = \vec{v_2}$

$\left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 8 & -6 & -3 & -2 \end{array}\right]$ $v_3 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

$y_1(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^0$ $y_2(t) = e^t \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$y_3(t) = e^t \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$w_1 = y'(t) = Aw_0 + b$ $Aw_1 = 0$

$\Rightarrow y_p(t) = w_0 + w_1 t$

$y(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t)$

$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -2 & 0 & 4 \end{array}\right) \Rightarrow \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \begin{matrix} c_1 \\ c_2 \\ c_3 \end{matrix}$

$y(t) = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - e^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + 2e^t \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + 2te^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

Q5 $y(t) \leftrightarrow Y(s)$ $f(t) \leftrightarrow F(s)$

$Y(s) = H(s)F(s)$

$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

$f(t) \to F(s)$ $f(at) \to \frac{1}{a} F(\frac{s}{a})$

$f(t) \leftrightarrow F(s)$ $t_0 > 0$ $f(t - t_0) \leftrightarrow F(s) e^{-st_0}$

$f(t) e^{s_0 t} \leftrightarrow F(s - s_0)$

$\delta(t) \leftrightarrow 1$ $e^{pt} u(t) \leftrightarrow \frac{1}{s-p}$ $te^{pt} u(t) \leftrightarrow \frac{1}{(s-p)^2}$

$t^n e^{pt} u(t) \leftrightarrow \frac{n!}{(s-p)^{n+1}}$ $\cos(\omega_0 t) u(t) \leftrightarrow \frac{s}{s^2 + \omega_0^2}$

$e^{-\alpha t} \cos(\omega_d t) u(t) \leftrightarrow \frac{s + \alpha}{(s+\alpha)^2 + \omega_d^2}$

$\delta'(t) \leftrightarrow s$ $u(t) \leftrightarrow \frac{1}{s}$ $tu(t) \leftrightarrow \frac{1}{s^2}$

$t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$ $\sin(\omega_0 t) u(t) \leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2}$

$e^{-\alpha t} \sin(\omega_d t) u(t) \leftrightarrow \frac{\omega_d}{(s+\alpha)^2 + \omega_d^2}$

例: $y'' + 4y = f(t)$ $y(0) = 0$ $y'(0) = 1$

$s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = F(s)$

$(s^2 + 4)Y(s) - 1 = F(s)$

$f(t) = t[u(t) - u(t-1)] + u(t-1) - u(t-3)$

$= tu(t) - (t-1)u(t-1) - u(t-3)$

$F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s}$

$Y(s) = \frac{F(s) + 1}{s^2 + 4}$ $- e^{-s}\left(\frac{1}{4s^2} - \frac{1}{4(s^2+4)}\right)$

$= \frac{1}{4s^2} + \frac{3}{4(s^2+4)} - e^{-3s}\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right)$

$y(t) = u(t)\left(\frac{t}{4} + \frac{3}{8}\sin 2t\right) - u(t-3)\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-3))\right) - u(t-1)\left(\frac{t-1}{4} - \frac{1}{8}\sin(2(t-1))\right)$

$[0, 1]$; $[1, 3]$ $[3, +\infty)$

$y_i(t) = \sum_{k=0}^{m_i - 1} \frac{1}{k!} t^k e^{\lambda_i t} (A - \lambda_i I_n)^k \vec{v_j}$

Q6 a) 先写特征方程解特征值

$a_1 \neq a_2$ $c_1 e^{a_1 t} + c_2 e^{a_2 t}$

$a_{1,2} = \alpha \pm i\beta$ $y = c_1 e^{\alpha t} \cos\beta t + c_2 e^{\alpha t} \sin\beta t$

$a_1 = a_2$ $y = c_1 e^{a_1 t} + c_2 t e^{a_1 t}$

b) general real solution = general solution + particular solution

$ke^{rx} \leftrightarrow Ce^{rx}$

$kx^n \leftrightarrow k_n x^n + k_{n-1} x^{n-1} + \dots + k_0$

$k\cos\omega x \leftrightarrow k\cos\omega x + M\sin\omega x$

$k\sin\omega x$

$ke^{\alpha x}\cos\omega x \leftrightarrow e^{\alpha x}(k\cos\omega x + M\sin\omega x)$

$ke^{\alpha x}\sin\omega x$

$y_p = axe^x$ 要化简 $y_p = Axe^x + Bx^2 e^x$

$e^{ix} = \cos x + i\sin x$

注意 y_p 和 y_h 有重根也要乘 t

例: a) $y''' + y'' - 2y = 0$

$x^3 + x^2 - 2 = 0$ $(x^2 + 2x + 2)(x - 1) = 0$

$\lambda_1 = -1 + i$ $\lambda_2 = -1 - i$ $\lambda_3 = 1$

e^t $e^{-t}\cos t$ $e^{-t}\sin t$

b) $y''' + y'' - 2y = 1 - 2t^3 + e^{-t}\cos t$

$y_{p1} = a$ $-2a = 1$ $a = -\frac{1}{2}$ (可以不用)

$y_{p2} = c_0 t^3 + c_1 t^2 + c_2 t + c_3$

$6c_0 + 2c_1 - 2c_3 + (6c_0 - 2c_2)t - 2c_1 t^2 - 2c_0 t^3 = 1 - 2t^3$

$c_3 = \frac{5}{2}$, $c_2 = 3$ $c_1 = 0$ $c_0 = 1$

$y_{p2} = \frac{5}{2} + 3t + t^3$

$y_{p3} = e^{-t}(k\cos t + M\sin t)$

$y_{p3}^* =$ ~~$e^{(-1+i)t}$~~ $cte^{(-1+i)t}$ 有重根 乘t

代入求解 $y_{p3} = -\frac{1}{10} te^{-t}\cos t - \frac{1}{5} te^{-t}\sin t$ 慢慢算

Q3 Table

$\int x^n dx = \frac{x^{n+1}}{n+1}$ $\quad \int \frac{1}{x} dx = \ln|x|$

$\int e^x dx = e^x$ $\quad \int b^x dx = \frac{b^x}{\ln b}$

$\int \sin x\, dx = -\cos x$ $\quad \int \cos x\, dx = \sin x$

$\int \sec^2 x\, dx = \tan x$ $\quad \int \csc^2 x\, dx = -\cot x$

$\int \sec x \tan x\, dx = \sec x$ $\quad \int \csc x \cot x\, dx = -\csc x$

$\int \sec x\, dx = \ln|\sec x + \tan x|$ $\quad \ln|\sec x|$

$\int \csc x\, dx = \ln|\csc x - \cot x|$ $\quad \int \tan x\, dx =$

$\int \cot x\, dx = \ln|\sin x|$ $\quad \int \sinh x\, dx = \cosh x$

$\int \cosh x\, dx = \sinh x$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ $\quad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$ $\quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right|$

$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta$ $\quad 1-\sin^2\theta = \cos^2\theta$

$\sqrt{a^2+x^2}$ $\quad x = a\tan\theta$ $\quad 1+\tan^2\theta = \sec^2\theta$

$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta$ $\quad \sec^2\theta - 1 = \tan^2\theta$

$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$\int_a^b f(x) g'(x) dx = f(x)g(x)\Big]_a^b - \int_a^b g(x) f'(x) dx$

$\int u\,dv = uv - \int v\,du$

maximal solution and domain
就是求解析式和定义域

例 $\quad y' = \frac{y}{t+1} + y^4 \quad t > -1$

$z(t) = y(t)^r \quad z' = r y^{r-1} y'$

$= r y^{r-1}\left(\frac{y}{t+1} + y^4\right)$

$z' = \frac{r}{t+1} y^r + r \cdot y^{r+3}$ $\quad$ let $r = -3$

$z(t) = \frac{1}{y(t)}$ $\quad z' = \frac{r}{t+1} z + r$

$z' = -\frac{3}{t+1} z - 3$

形如 $y' + p(x)y = q(x)$ $\quad$ 记得补常数
$u = e^{\int p(x) dx}$ $\quad$ 同乘u 积分

$z' + \frac{3}{t+1} z = -3 \quad u = e^{\int \frac{3}{t+1} dt} = (t+1)^3$ Q1

$(t+1)^3 z = -\frac{3}{4}(t+1)^4 + C$

$z = -\frac{3}{4}(t+1) + \frac{C}{(t+1)^3}$

$y(t) = (t+1)\left(C - \frac{3}{4}(t+1)^4\right)^{-\frac{1}{3}}$

for $C \le 0$ or $C = \infty$ $\quad C \in \mathbb{R} \cup \{\infty\}$

maximal solution have domain $(-1, \infty)$

for $C > 0$ $\quad \left(-1, \left(\frac{4C}{3}\right)^{\frac{1}{4}} - 1\right) \cup \left(\left(\frac{4C}{3}\right)^{\frac{1}{4}} - 1, \infty\right)$

+题 $\quad F(x,y) = \frac{1}{2}x^2 + \frac{1}{6}\ln|6y^2 - 1|$ $\quad F_{xy} - F_{yx} = 0$

$dF(x,y) = x\,dx + \frac{2y}{6y^2-1} dy \quad F_{xx} = 1$

$F_x(0,0) = 0 \quad F_y(0,0) = 0$ $\quad$ saddle point

$D = F_{xx} F_{yy} - [F_{xy}]^2 = -1 < 0$

hence contained in two distinct integral curves.

exact equation $\quad M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (需满足) $\quad$ 积分因子 $\mu(s)$

$\mu(x) \quad \frac{M_y - N_x}{N} = g(x)$ — ①

$\mu(y) \quad \frac{M_y - N_x}{M} = g(y)$ — ②

$\mu(xy) \quad \frac{M_y - N_x}{Ny - Mx} = g(xy)$ — ③

$\mu\left(\frac{y}{x}\right) \quad \frac{x^2(M_y - N_x)}{Ny + Mx} = g\left(\frac{y}{x}\right)$ — ④

①③ $\mu'(s) = g(s)\mu(s)$ $\quad$ ②④ $\mu'(s) = -g(s)\mu(s)$

例: $(3xy + 2y^2)dx + (3x^2 + 6xy + 3y^3)dy = 0$

($M = 3xy + 2y^2$, $N = 3x^2 + 6xy + 3y^3$)

$M_y = 3x + 4y \quad N_x = 6x + 6y \quad M_y - N_x = -3x - 2y$

$\frac{M_y - N_x}{M} = -\frac{1}{y} \quad \mu' = \frac{1}{y}\mu \quad \mu = y.$

$\Rightarrow (3xy^2 + 2y^3)dx + (3x^2y + 6xy^2 + 3y^4)dy = 0$

$f(x,y) = \frac{3}{2}x^2y^2 + 2xy^3 + g(y)$ $\quad + y^3$

$f_y(x,y) = 3x^2y + 6xy^2 + g'(y) = 3x^2y + 6xy^2 + 3y^3$

$g'(y) = 3y^3 \quad g(y) = \frac{3}{4}y^4 + C$

$\Rightarrow 6x^2y^2 + 8xy^3 + 3y^4 = C \quad C \in \mathbb{R}$

$C \ge 0$
需要判断一下取值范围