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Tusnady's inequality revisited

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Abstract

Tusnady's inequality is the key ingredient in the KMT/Hungarian coupling of the empirical distribution function with a Brownian bridge. We present an elementary proof of a result that sharpens the Tusnady inequality, modulo constants. Our method uses the beta integral representation of Binomial tails, simple Taylor expansion and some novel bounds for the ratios of normal tail probabilities.

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TUSNÁDY'S INEQUALITY REVISITED

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Tusnády's inequality is the key ingredient in the KMT/Hungarian coupling of the empirical distribution function with a Brownian bridge. We present an elementary proof of a result that sharpens the Tusnády inequality, modulo constants. Our method uses the beta integral representation of Binomial tails, simple Taylor expansion and some novel bounds for the ratios of normal tail probabilities.

1. Introduction. In one of the most important probability papers of the last forty years, Komlós, Major and Tusnády (1975) sketched a proof for a very tight coupling of the standardized empirical distribution function with a Brownian bridge, a result now often referred to as the KMT, or Hungarian, construction. Their coupling greatly simplifies the derivation of many classical statistical results—see Shorack and Wellner [(1986), Chapter 12 et seq.], for example.

The construction has taken on added significance for statistics with its use by Nussbaum (1996) in establishing asymptotic equivalence of density estimation and white noise models. Brown, Carter, Low and Zhang (2004) have somewhat simplified and expanded Nussbaum's argument using our Theorem 2, via inequality (5).

At the heart of the KMT method [with refinements as in the exposition by Csörgő and Révész (1981), Section 4.4] lies the quantile coupling of the $\text{Bin}(n, 1/2)$ and $N(n/2, n/4)$ distributions, which may be defined as follows. Let Y be a random variable distributed $N(n/2, n/4)$. Find the cutpoints $-\infty = \beta_0 < \beta_1 < \cdots < \beta_n < \beta_{n+1} = \infty$ for which

$$\mathbb{P}\{\text{Bin}(n, 1/2) \geq k\} = \mathbb{P}\{Y > \beta_k\} \qquad \text{for } k = 0, 1, \ldots, n.$$

When $\beta_k < Y \leq \beta_{k+1}$, let X take the value k. Then X has a $\text{Bin}(n, 1/2)$ distribution.

It is often more convenient to work with the tails of the standard normal $\bar{\Phi}(z) = \mathbb{P}\{N(0,1) > z\}$ and the standardized cutpoint $z_k = 2(\beta_k - n/2)/\sqrt{n}$, thereby replacing $\mathbb{P}\{Y > \beta_k\}$ by $\bar{\Phi}(z_k)$.

Symmetry considerations show that $\beta_{n-k+1} = n - \beta_k$, so that it suffices to consider only half the range for k. More precisely, when n is even, say $n = 2m$,

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