Arc Length and Riemannian Metric Geometry

References:

1. W. F. Reynolds, Hyperbolic geometry on a hyperboloid, *Amer. Math. Monthly* **100** (1993) 442–455.

2. Wikipedia page "Metric tensor". The most pertinent parts are at the beginning and end. (Beware that there are other, less relevant, pages with "metric" in their titles.)

3. Greenberg, Appendix A, Riemannian part.

4. In the long run you might want to take Math. 439, "Differential Geometry of Curves and Surfaces", or Math. 460. "Tensors and General Relativity".

We shall look at a sequence of increasingly general or complicated siutations.

1. Length of the graph of a function

The length of a short chord segment is given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2.$$

Drop the parentheses.

$$s \equiv \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \Delta s_i = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$= \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \Delta x_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} = \int_{x_0}^{x_f} \sqrt{1 + f'(x)^2}\, dx.$$

Shorthand: $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + f'(x)^2}\, dx.$

2. Length of a parametrized curve

Let $x = x(t)$, $y = y(t)$. $\Delta s^2 = \Delta x^2 + \Delta y^2$ still.

$$s = \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$= \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} \Delta t_i \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2} = \int_{t_0}^{t_f} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Shorthand: $ds = \sqrt{dx^2 + dy^2} = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$ (where $\dot{x} \equiv \frac{dx}{dt}$, etc.).

Special case: $t = x$. Then $\dot{x} = 1$, $ds = \sqrt{1 + \dot{y}^2}\, dx$ — same as situation 1.

3. Three dimensions

Let $x = x(t)$, $y = y(t)$, $z = z(t)$. Then

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad s = \int_{t_0}^{t_f} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt.$$

4. Polar coordinates in the plane

Define r and θ by $x = r\cos\theta$, $y = r\sin\theta$. A change $\Delta\theta$ changes lengths by $r\Delta\theta$. Therefore, it is intuitively clear (and we'll justify it in a moment) that

$$ds^2 = dr^2 + r^2 d\theta^2.$$

Example: Consider the arc at $r = 2$ from $\theta = 0$ to $\theta = \pi/4$. Let θ play the role of the parameter, t.

$$ds = \sqrt{dr^2 + r^2 d\theta^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + 2^2\left(\frac{d\theta}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{0 + 4}\, d\theta = 2\, d\theta.$$

$$s = \int_0^{\pi/4} 2\,d\theta = \frac{\pi}{2}\,.$$

In general,

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\,dt.$$

(Note that r in the second term is a function of t, in general.)

A shorthand calculation leading to the arc length formula: Calculate

$$\begin{aligned} dx &= \cos\theta\,dr - r\sin\theta\,d\theta, \\ dy &= \sin\theta\,dr + r\cos\theta\,d\theta. \end{aligned}$$

(This really means that

$$\Delta x = \cos\theta\,\Delta r - r\sin\theta\,\Delta\theta + \text{something very small},$$

etc.) It follows that

$$\begin{aligned} dx^2 &= \cos^2\theta\,dr^2 + r^2\sin^2\theta\,d\theta^2 - 2r\sin\theta\cos\theta\,dr\,d\theta, \\ dy^2 &= \sin^2\theta\,dr^2 + r^2\cos^2\theta\,d\theta^2 + 2r\sin\theta\cos\theta\,dr\,d\theta. \end{aligned}$$

Therefore, $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$.

5. A curved surface embedded in 3-space

Example: The unit sphere, $x^2 + y^2 + z^2 = 1$. As usual let

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

Then, for instance, we have

$$dz = \cos\theta\,dr - r\sin\theta\,d\theta,$$

but r is a constant in our problem ($r = 1$), so we can set $dr = 0$. Thus, ignoring dr, we have

$$\begin{aligned} dx &= \cos\theta\cos\phi\,d\theta - \sin\theta\sin\phi\,d\phi, \\ dy &= \cos\theta\sin\phi\,d\theta + \sin\theta\cos\phi\,d\phi. \end{aligned}$$

It easily follows that

$$ds^2 = dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2\theta\,d\phi^2.$$

Note that near the "north pole", $\theta = 0$, we have $\sin\theta \approx \theta$ and hence $ds^2 \approx d\theta^2 + \theta^2\,d\phi^2$. That is, near the pole the angular spherical coordinates "look like" plane polar coordinates, with θ in the role of r and ϕ in the role of θ.