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16.07 Dynamics
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Lecture L14 - Variable Mass Systems: The Rocket Equation

In this lecture, we consider the problem in which the mass of the body changes during the motion, that is, m is a function of t, i.e. $m(t)$. Although there are many cases for which this particular model is applicable, one of obvious importance to us are rockets. We shall see that a significant fraction of the mass of a rocket is the fuel, which is expelled during flight at a high velocity and thus, provides the propulsive force for the rocket.

As a simple model for this process, consider the cases sketched in a) through f) of the figure. In a) we consider 2 children standing on a stationary flat car. At $t = 0$, they both jump off with velocity **relative to the flat car** of u. (While this may not be a good model for children jumping off a flatcar, it is a good model for the expulsion of mass from a rocket where the mass flow from the choked nozzle occurs at a relative velocity u from the moving rocket.) In b) we consider that they jump off in sequence, each jumping with a velocity u relative to the then velocity of the flatcar, which will be different for the 2nd jumper since the car has begun to move as a result of jumper 1. For a), we have a final velocity of the flat car as

$$V_{2R} = \frac{2mu}{M + 2m} \tag{1}$$

for b) we have

$$V_{22} = \frac{mu}{M + 2m} + \frac{mu}{M + m} \tag{2}$$

where M is the mass of the flatcar, and m is the mass of the jumper. We can see the case b) gives a higher final velocity. We now consider cases with more blocks; we introduce the notation V_{NN} to be the final velocity with N blocks coming off one at a time. We use the notation V_{NR} to denote the reference velocity if the N blocks come off together. If we consider 3 blocks, we can see by inspection that

$$V3R = \frac{3mu}{M + 3m} \tag{3}$$

and

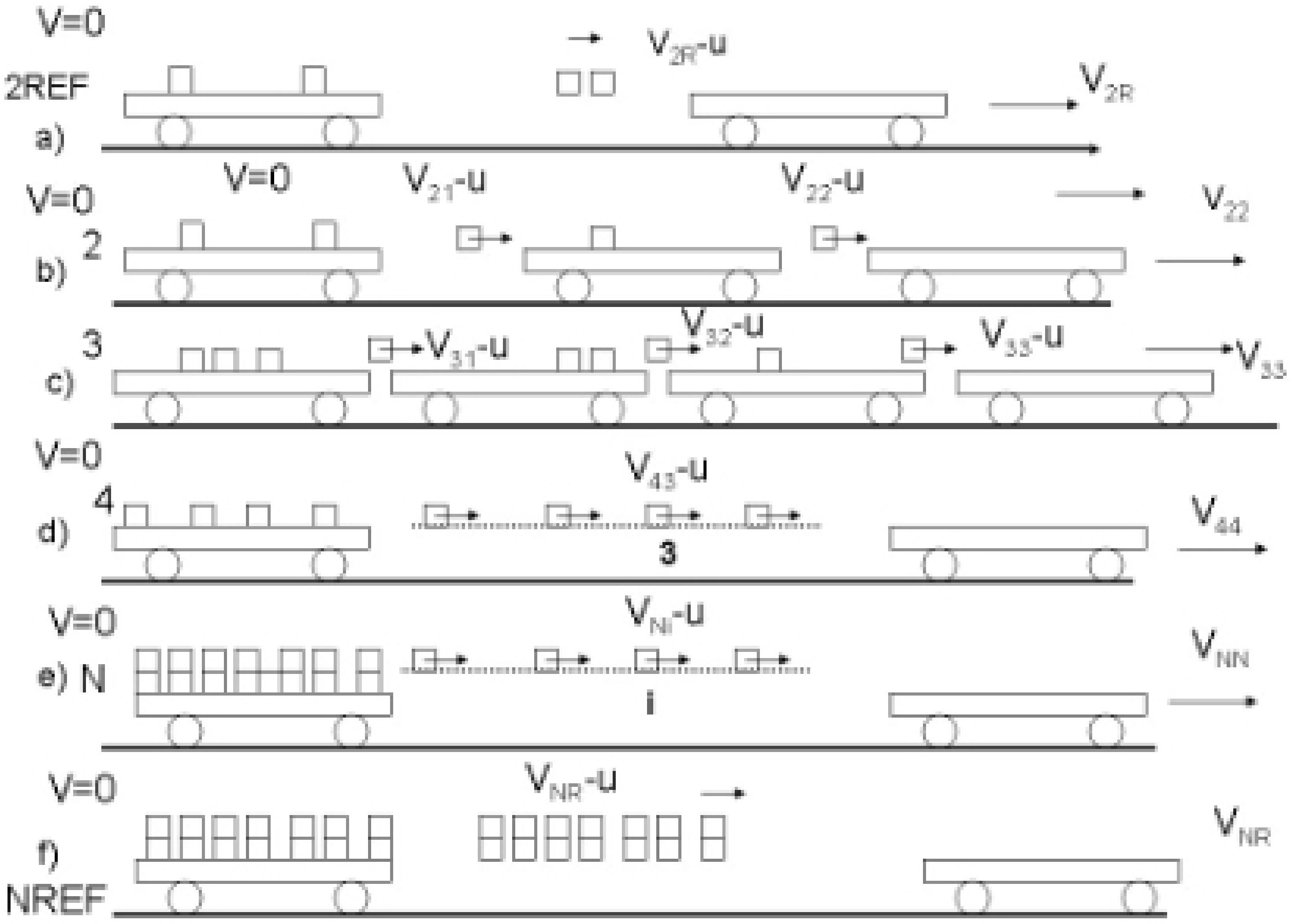
$$V_{33} = \frac{mu}{M + 3m} + \frac{mu}{M + 2m} + \frac{mu}{M + m} \tag{4}$$

While for 4 blocks

$$V_{4R} = \frac{4mu}{M + 4m} \tag{5}$$

and

$$V_{44} = \frac{mu}{M + 4m} + \frac{mu}{M + 3m} + \frac{mu}{M + 2m} + \frac{mu}{M + m} \tag{6}$$

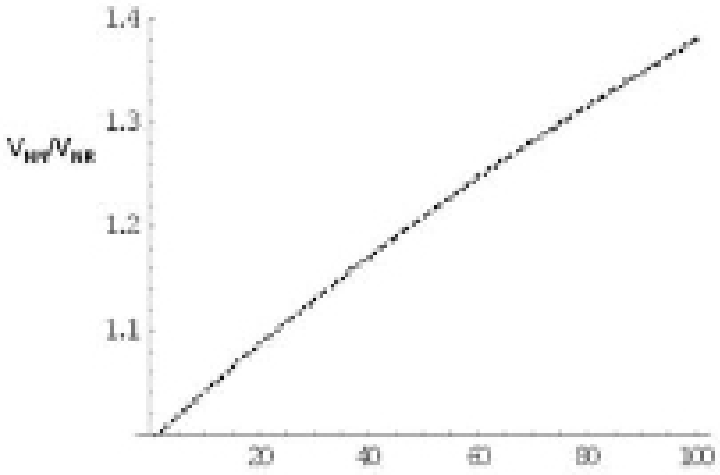


Generalizing this to arbitrary N, we can write

$$V_{NR} = \frac{Nmu}{M + Nm} \tag{7}$$

and

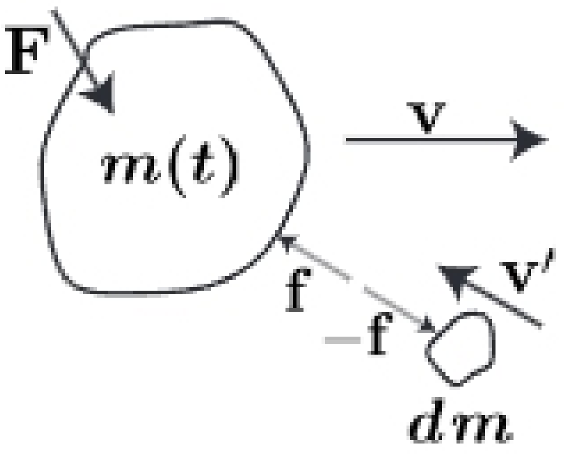
$$V_{NN} = \sum_{i=1}^{N} \frac{mu}{M + im} \tag{8}$$



The figure shows the results for $V_{100,100}/V_{100R}$, the increase in velocity as each block comes off. We choose 100 blocks with a mass m equal to .01 M; therefore the 100 blocks with $m = .01$ equal the mass M, the propellant ratio of the "rocket" is 50%. The final velocity of the cart is 40% higher than would be obtained if the masses come off together. Momentum is conserved. The sum of the momentum of the cart plus all the individual mass points is equal to zero. However, the momentum of all the particles left behind is of very little interest to us. We care only about the final velocity of the cart.

To analyze this question we must consider a system of variable mass, and the process by which it gains velocity as a result of ejecting mass. While our previous discussion provided a discrete model of this process, we now consider a continuous process, obtaining a relation for $v(t)$ as a function of the momentum interaction between the system and the external world.

We shall start by considering a body with velocity $\boldsymbol{v}$ and external forces $\boldsymbol{F}$, gaining mass at a rate $\dot{m} = dm/dt$. Let us look at the process of gaining a small amount of mass dm. Let $\boldsymbol{v}'$ be the velocity of dm before it is captured by m, and let $\boldsymbol{f}$ represent the average value of the impulsive forces that dm exerts on m during the short interval dt, in which the capturing takes place. By Newton's third law, dm will experience a force $-\boldsymbol{f}$, exerted by m, over the same dt.



We can now examine the capture process from the point of view of dm and equate the impulse, $-\boldsymbol{f}dt$, to the change in linear momentum of dm,

$$-\boldsymbol{f}dt = dm(\boldsymbol{v} + d\boldsymbol{v} - \boldsymbol{v}'). \tag{9}$$

Here, $\boldsymbol{v} + d\boldsymbol{v}$ is the velocity of m (and dm) after impact. Analogously, from the point of view of m, we write

$$\boldsymbol{F}dt + \boldsymbol{f}dt = m(\boldsymbol{v} + d\boldsymbol{v}) - m\boldsymbol{v} = m d\boldsymbol{v}. \tag{10}$$

The term $dm\, d\boldsymbol{v}$ in equation (9) is a higher order term and will disappear when we take limits. The impulse due to the contact force can be eliminated by combining equations (9) and (10),

$$\boldsymbol{F}dt - dm(\boldsymbol{v} - \boldsymbol{v}') = m d\boldsymbol{v},$$

or, dividing through by dt,

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} - (\boldsymbol{v} - \boldsymbol{v}')\frac{dm}{dt} = \boldsymbol{F} + (\boldsymbol{v}' - \boldsymbol{v})\frac{dm}{dt}. \tag{11}$$

Here, $\boldsymbol{v}' - \boldsymbol{v}$ is the velocity of dm relative to m. This expression is valid when $dm/dt > 0$ (mass gain) and when $dm/dt < 0$ (mass loss). If we compare this expression to the more familiar form of Newton's law for a particle of fixed mass $m\frac{dv}{dt} = F$, we see that the term $(\boldsymbol{v}' - \boldsymbol{v})dm/dt$ is an additional force on m which is due to the gain (or loss) of mass. Equation (11) can also be written as

$$\frac{d(m\boldsymbol{v})}{dt} = \boldsymbol{F} + \boldsymbol{v}'\frac{dm}{dt},$$