MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
8.01 **Fall 2010**

Math Review Module: Work and the Scalar Product

4.1 Scalar Product (Dot Product)

We shall introduce a vector operation, called the "dot product" or "scalar product" that takes any two vectors and generates a scalar quantity (a number). We shall see that the physical concept of work can be mathematically described by the dot product between the force and the displacement vectors.

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Because any two non-collinear vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure 4.1. Note that θ can vary from 0 to π.

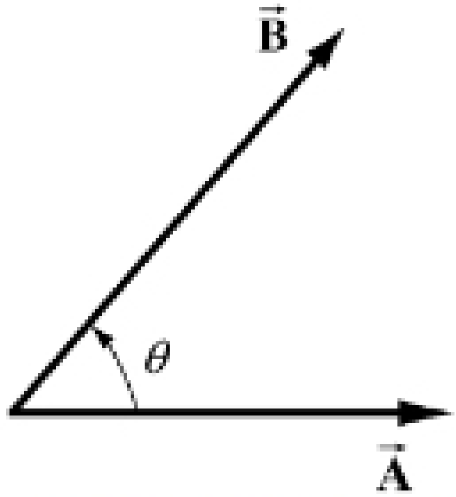

Figure 4.1 **Dot product geometry.**

Definition: **Dot Product**

The dot product $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the cosine of the angle θ between the two vectors:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos(\theta) \tag{4.1}$$

Where $A = |\vec{\mathbf{A}}|$ and $B = |\vec{\mathbf{B}}|$ represent the magnitude of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ respectively. The dot product can be positive, zero, or negative, depending on the value of $\cos\theta$. The dot product is always a scalar quantity.

The angle formed by two vectors is therefore

$$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|}\right) \quad (4.2)$$

The magnitude of a vector $\vec{\mathbf{A}}$ is given by the square root of the dot product of the vector $\vec{\mathbf{A}}$ with itself.

$$|\vec{\mathbf{A}}| = (\vec{\mathbf{A}} \cdot \vec{\mathbf{A}})^{1/2} \quad (4.3)$$

We can give a geometric interpretation to the dot product by writing the definition as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A\cos(\theta))\,B \quad (4.4)$$

In this formulation, the term $A\cos\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{B}}$. This projection is shown in Figure 4.2(a). So the dot product is the product of the projection of the length of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$ with the length of $\vec{\mathbf{B}}$. Note that we could also write the dot product as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A(B\cos(\theta)) \quad (4.5)$$

Now the term $B\cos(\theta)$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{A}}$ as shown in Figure 4.2(b). From this perspective, the dot product is the product of the projection of the length of $\vec{\mathbf{B}}$ in the direction of $\vec{\mathbf{A}}$ with the length of $\vec{\mathbf{A}}$.

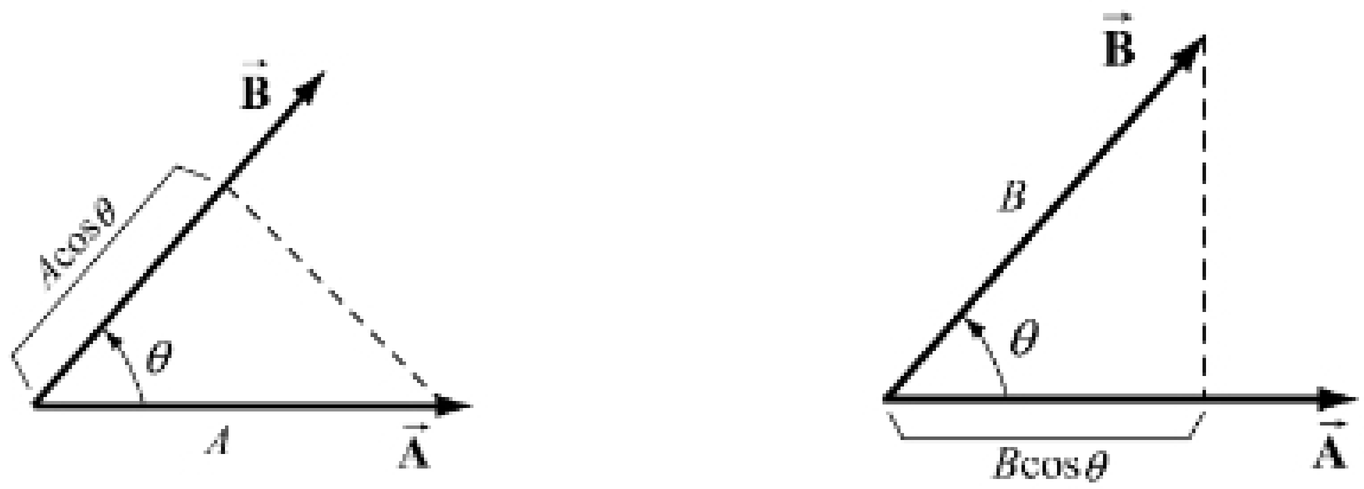

Figure 4.2(a) **and** 4.2(b) **Projection of vectors and the dot product.**

From our definition of the dot product we see that the dot product of two vectors that are perpendicular to each other is zero since the angle between the vectors is $\pi / 2$ and $\cos(\pi / 2) = 0$.

We can calculate the dot product between two vectors in a Cartesian coordinates system as follows. Consider two vectors $\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}$

Recall that

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$$
$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = 0 \quad (4.6)$$

The dot product between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is then

$$\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z \quad (4.7)$$

The time derivative of the dot product of two vectors is given by

$$\begin{aligned}\frac{d}{dt}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}) &= \frac{d}{dt}(A_x B_x + A_y B_y + A_z B_z)\\ &= \frac{d}{dt}(A_x)B_x + \frac{d}{dt}(A_y)B_y + \frac{d}{dt}(A_z)B_z + A_x\frac{d}{dt}(B_x) + A_y\frac{d}{dt}(B_y) + A_z\frac{d}{dt}(B_z),\\ &= \left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{B}} + \vec{\mathbf{A}}\cdot\left(\frac{d}{dt}\vec{\mathbf{B}}\right)\end{aligned} \quad (4.8)$$

In particular when $\vec{\mathbf{A}} = \vec{\mathbf{B}}$, then the time derivative of the square of the magnitude of the vector $\vec{\mathbf{A}}$ is given by

$$\frac{d}{dt}A^2 = \frac{d}{dt}(\vec{\mathbf{A}}\cdot\vec{\mathbf{A}}) = \left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{A}} + \vec{\mathbf{A}}\cdot\left(\frac{d}{dt}\vec{\mathbf{A}}\right) = 2\left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{A}}, \quad (4.9)$$

4.2 Kinetic Energy and the Dot Product

For an object undergoing three-dimensional motion, the velocity of the object in Cartesian components is given by $\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$. Recall that the magnitude of a vector is given by the square root fo the dot product of the vector with itself,

$$A \equiv \left|\vec{\mathbf{A}}\right| \equiv (\vec{\mathbf{A}}\cdot\vec{\mathbf{A}})^{1/2} = (A_x^2 + A_y^2 + A_z^2)^{1/2} \quad (4.10)$$

Therefore the square of the magnitude of the velocity is given by the expression

$$v^2 \equiv (\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}) = v_x^2 + v_y^2 + v_z^2 \quad (4.11)$$

Hence the kinetic energy of the object is given by

$$K = \frac{1}{2}m(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}) = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) \quad (4.12)$$