# ME 201/MTH 281/ME 400/CHE 400
# ASSIGNMENT #2 2011

Assignments handed in by 6 PM on Wednesday September 14 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday September 15 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday.

In problems (3) and (4) and in the challenge problem, you will need to calculate Fourier coefficients. In doing that, you can save considerable labor by using Mathematica. The last page of this assignment gives an example of such a calculation.

## LECTURE SCHEDULE AND READING

| Section in Class Notes | Date | Section in Text |
|---|---|---|
| 1.3 Separation of Variables – A First Attempt | F,W Sept. 2,7 | 2.1, 2.2, 2.3.1-2.3.5 |
| 2.1 Basic Fourier Series | Th,F,M Sept. 8,9,12 | 3.1 |

## PROBLEMS

### 1.3 SEPARATION OF VARIABLES

**(1)** (20 points) Use the separated solutions found in class to find an explicit solution to the boundary value problem given below.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2},\ T(0,t)=0,\ T(L,t)=0, \text{ and}$$

$$T(x,0) = 4T_0 \sin\left(\frac{\pi x}{L}\right)\cos^3\left(\frac{\pi x}{L}\right),$$

where $T_0$ is a constant. (Hint: You will need to use some standard trig identities to transform the initial condition.)

**(2)** (30 points) Consider the boundary value problem given below for heat conduction in a bar.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2},\ 0<x<L,\ t>0, \text{ with } \frac{\partial T}{\partial x}(0,t)=0,\ \frac{\partial T}{\partial x}(L,t)=0, \text{ and } T(x,0)=f(x).$$

This is the problem for a bar initially at temperature $f(x)$, with both the left and right ends perfectly insulated.

**(a)** (15 points) Find all of the separated solutions satisfying the equation and the two homogeneous boundary conditions. The calculation will be similar to the one done in section 1.3 of the class notes.

**(b)** (5 points) Find an explicit solution to the boundary value problem when $f(x) = T_1\cos\left(\frac{\pi x}{L}\right) + T_2\cos\left(\frac{2\pi x}{L}\right)$, where $T_1$ and $T_2$ are constants.

**(c)** (5 points) Find an explicit solution to the boundary value problem when $f(x) = T_0 + T_1\cos\left(\frac{\pi x}{L}\right)$, where $T_0$ and $T_1$ are constants. (Hint: If you have difficulty with the $T_0$ term, go back to part (a) and make sure that you find <u>all</u> of the solutions.)

**(d)** (5 points) Try to solve the problem when $f(x) = T_0\frac{x}{L}$, where $T_0$ is a constant. You will not be able to finish this problem at the present point of the course. Discuss this situation and identify the problems which keep you from finishing the calculation.

**(CONTINUED NEXT PAGE)**

## 2.1 BASIC FOURIER SERIES

**(3)** (25 points) Consider the function $f(x)$ defined by $f(x) = 1 + x|x|$ for $-1 \le x < 1$, and $f(x+2) = f(x)$ for all $x$.

**(a)** (5 points) Give a sketch of $f(x)$ versus $x$ for $-3 \le x \le 3$.

**(b)** (5 points) Give the values of all the following:
$f(-1), f(0), \lim_{x\to 1-} f(x), \lim_{x\to 1+} f(x), f(1), f(6)$, and $f(7)$.

**(c)** (5 points) Is $f$ piecewise smooth? (Justify your answer.)

**(d)** (5 points) Find the Fourier series for $f$.

**(e)** (5 points) Using the basic convergence theorem, tell what the series converges to for each $x$ in [-1, 1].

## COMPUTER WORK

For this assignment, you are asked to work through the following five sections of the Mathematica Tutorial: Arithmetic; Algebra; Defining Constants, Expressions and Functions; Plotting; Summing Series. You may download the tutorial from the course web site. After you have loaded the tutorial notebook into Mathematica, you can open any of the sections by double clicking on the square bracket opposite the section name. You do not need to hand in any of your work with the tutorial.

**(4)** (25 points) In this problem, you will continue working with the series of problem 3.

**(a)** (5 points) Use Mathematica to graph the function $f$ over the interval [-3, 3]. (Hint: If you have defined a function fbas[x] over the interval $[-L, L)$, the extension of fbas which is periodic with period $2L$ can be defined in Mathematica by f[x_] := fbas[Mod[x, 2*L, -L] .)

**(b)** (20 points) Use the Sum function, which you worked with in the tutorial, to define a function which plots the $M^{th}$ partial sum of the series. By Mth partial sum, the following is meant:

$$a_0 + \sum_{n=1}^{M} \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\}.$$

Your function should take $M$ as an argument, and produce a graph of the above expression over the $x$-range [-3,3]. Use your function to produce three plots of partial sums for M = 5, 10 and 20, and for $x$ in the interval [-3, 3]. Show the original function $f$ on each of your three graphs. Do you think 20 terms is enough for the series to represent the original function? (You may either answer that question or criticize it.)

## CHALLENGE PROBLEM

In this problem, you will look at the rate of convergence of Fourier series. As an example, consider the series discussed in class for $f_{bas}(x) = x^2$ on the interval $[-L, L]$. The Fourier coefficients were given in class: $a_0 = \frac{L^2}{3}$, $a_n = \frac{4L^2(-1)^n}{n^2\pi^2}$, and $b_n = 0$. In this case, we say that the Fourier coefficients drop off like $1/n^2$, and this is what we mean by the rate of convergence. It is of considerable practical importance. If the convergence is like $1/n$, as it is for the square wave, we need on the order of 100 terms to reduce the error to 1%. If the convergence is like $1/n^2$ as it is in the present example, we need only on the order of 10 terms to reduce the error to 1%. (Note that these are guidelines and not exact statements.) In this problem, you will look at the Fourier series for three different functions, and on the basis of your findings, you will be asked to speculate about the relation between the periodically extended function and the rate of convergence. You can save a lot of labor by using Mathematica to calculate the Fourier coefficients. Examples of that are given on the last page of this assignment. For each of the four functions given below on the base interval [-1, 1], plot the extended periodic function on [-3, 3], find the Fourier coefficients, and state the convergence rate (like $1/n$, or $1/n^2$, or $1/n^3$, . . . .). You should find four different rates of convergence for your four examples. Look at your graphs carefully and speculate on what differences in the extended functions might account for the different rates of convergence. This is a topic we will cover in detail a little later in the course.

(a) $f_{bar}(x) = x$,

(b) $f_{bar}(x) = x^2$,

(c) $f_{bar}(x) = x - x^3$,

(d) $f_{bar}(x) = x^4 - 2x^2$.

# ME 201/MTH 281/ME 400/CHE 400 Example of Calculation of Fourier Coefficients

We show in this notebook how to use *Mathematica* to calculate the Fourier coefficients of a given functi

We take the base function to be

In[1]:= `f[x_] := x + x^2`

and the base interval to be $-L$ to $L$, with $L = 2$.

In[2]:= `L = 2;`

We let a[n] be the nth cosine coefficient, and b[n] be the nth sine coefficient. Then

In[3]:= $a[0] = \frac{1}{2L}\int_{-L}^{L} f[x]\, dx$

Out[3]= $\frac{4}{3}$

For n > 0, we calculate the Fourier cosine coefficient, which we temporarily call coeff:

In[4]:= $coeff = \frac{1}{L}\int_{-L}^{L} f[x] \, Cos[(n \pi x)/L]\, dx$

Out[4]= $\frac{8\left(2 n \pi Cos[n \pi] + \left(-2 + n^2 \pi^2\right) Sin[n \pi]\right)}{n^3 \pi^3}$