ME 201/MTH 281/ME 400/CHE 400 EXAM #2

THURSDAY NOVEMBER 10, 2011 2:00 – 3:15 PM and 3:25 – 4:40 PM

This exam covers homework assignments 5 through 8, and the following sections of the class notes: sections 3.2 through 3.6 of Chapter 3, Chapters 4 and 5, and sections 6.1 and 6.2 of Chapter 6. You may use any books, notes or reference material that you like, but you may not exchange reference material with anyone else. If you need additional information to work a problem, ask me for it, and, if it is appropriate, I will put it on the board. Do all four problems. The value of each is shown, and the total possible is 100. **BE SURE TO EXPLAIN YOUR WORK!** Wrong calculations with no explanation will receive very little partial credit. Solutions will be posted on the web when both exams are over. Graded exams will be returned in class on Wednesday. Good luck!

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(1) (25 points) Consider the regular Sturm-Liouville system given below.

$$\frac{d^2\psi}{dx^2} + \lambda x\psi - 3x\psi = 0,\ 1 < x < 2,\ \text{with } \frac{d\psi}{dx}(1) = 0 \text{ and } \psi(2) = 0.$$

(a) (10 points) Show that the eigenvalues are all greater than 3.

(b) (5 points) Let ψ_1 and ψ_2 be eigenfunctions associated with distinct eigenvalues. State the orthogonality condition satisfied by ψ_1 and ψ_2.

(c) (10 points) Give an example of a suitable trial function for estimating the first eigenvalue by using the Rayleigh quotient.

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(2) (40 points) Consider the boundary value problem given below for the Laplace-like equation in a rectangle. Here $f(x)$ is a given function, continuous and piecewise smooth on the upper boundary.

$$4\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \quad , \quad 0 < x < a \text{ and } 0 < y < b\ ,$$

$$\text{with } \Phi(0,y) = 0\ , \frac{\partial\Phi}{\partial x}(a,y) = 0\ , \frac{\partial\Phi}{\partial y}(x,0) = 0\ , \text{and } \Phi(x,b) = f(x).$$

(a) (15 points) Which of the expansions given below has the appropriate form for the solution of this problem? Explain your answer in detail.

$$(1)\ \Phi(x,y) = \sum_{n=1}^{\infty} F_n(y)\sin\left[(n-\tfrac{1}{2})\frac{\pi x}{a}\right],$$

$$(2)\ \Phi(x,y) = \sum_{n=1}^{\infty} G_n(y)\cos\left[(n-\tfrac{1}{2})\frac{\pi x}{a}\right],$$

$$(3)\ \Phi(x,y) = \sum_{n=1}^{\infty} H_n(x)\sin\left[\frac{n\pi y}{b}\right],$$

$$(4)\ \Phi(x,y) = K_0(y) + \sum_{n=1}^{\infty} K_n(y)\cos\left[\frac{n\pi x}{a}\right].$$

(2) (b) (25 points) By using the correct expansion from part (a), solve the problem when $f(x) = \alpha\sin\left(\frac{\pi x}{2a}\right) + \beta\sin\left(\frac{5\pi x}{2a}\right)$, where α and β are constants.

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(3) (25 points) As we showed in class, the partial differential equation governing the pressure variations p' in a sound field, depending only on the coordinate *x* and the time *t*, is the wave equation

$$\frac{\partial^2 p'}{\partial t^2} = C_0^2 \frac{\partial^2 p'}{\partial x^2},$$

where C_0 is the sound speed. Consider standing sound waves in an organ pipe of length *L* which is open at both the ends x = 0 and x = *L*. In this case the boundary conditions are that $p' = 0$ at both 0 and *L*. By looking for standing wave solutions (another name for normal modes) of angular frequency ω, find the musical pitch in Hz for this organ pipe in terms of *L* and C_0. Give all of the analysis leading to this result.

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(4) (10 points) Use the Table of Fourier Transforms (handed out in class earlier and also given in part below) to find the inverse transform of e^{-bk^2}. Here *b* is a positive constant.

Brief Table of Fourier Transforms

In this table, the Fourier transform and inverse transform are defined by the integrals

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx \text{ and } f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)dk.$$

f(x)	$\tilde{\mathbf{f}}(\mathbf{k})$
$\frac{1}{x^2+a^2}$, a > 0	$\frac{\pi}{a}e^{-a\|k\|}$
$e^{-a\|x\|}$, a > 0	$\frac{2a}{a^2+k^2}$
$xe^{-a\|x\|}$	$\frac{-4aik}{(a^2+k^2)^2}$
$x^2e^{-a\|x\|}$	$\frac{4a(a^2-3k^2)}{(a^2+k^2)^3}$
e^{-ax^2}, a > 0	$\sqrt{\frac{\pi}{a}}e^{-\frac{k^2}{4a}}$
xe^{-ax^2}, a > 0	$\frac{-ik\sqrt{\pi}}{2a^{3/2}}e^{-\frac{k^2}{4a}}$