II. Introduction to probability, 2

1 Random Variables

1.1 Definition:

A random variable is a function defined on a sample space. In other words, it is a mapping of events to numbers.

1.2 This can be a simple one-to-one mapping...

1.2.1 Example:

Throw a six-sided die, and let the random variable X be the face value.

1.3 ...or any specified function.

1.3.1 Example:

In our previous 8-point sample space of the outcomes of three coin tosses, we can define a random variable X as the number of heads. We then have:

Event	X
HHH	3
HHT	2
HTH	2
HTT	1
TTT	0
TTH	1
THT	1
THH	2

1.4 Random variables and probabilites (illustrate with sample space of discrete points):

Let $x_1, x_2, ...$ be all the values that the random variable X can take on. Then we denote the probability that X takes on the value x_j as $P(X = x_j) = f(x_j)$. In the previous example of the die, assuming it is fair, we have:

x_j	$f(x_j)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

And in the example of the sum of heads in three coin tosses, we have:

x_j	$f(x_j)$
0	1/8
1	3/8
2	3/8
3	1/8

2 Density and Distribution

2.1 Density:

The density function $f(x)$ is proportional to the probability that a random variable will take on a value between x and $x + \delta x$, where δx is an infinitesimal increment. (Strictly speaking, with a continuous distribution, the probability of taking on exactly a particular value is vanishingly small.)

2.1.1 By definition:

$\int_{-\infty}^{\infty} f(x)\, dx = 1$ if $f(x)$ is a density function.

2.2 Distribution:

The distribution function $F(x)$ is the probability that the random variable will take on a value less than or equal to x.

2.2.1

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$, and $f(x) = \frac{d[F(x)]}{dx}$

So, if we know the density we can determine the distribution function, and vice versa.

2.3 Discrete case:

$f(x_j) = P(X = x_j)$ is the *probability distribution*, and $F(x) = P(X \leq x) = \sum_{x_j \leq x} f(x_j)$ is the distribution function.

2.4 Examples

2.4.1 Exponential with parameter r

density: $f(x) = re^{-rx}$

distribution: $F(x) = 1 - e^{-rx}$