2.4 Linear Equations II

Studied here are the subjects of variation of parameters and undetermined coefficients for linear first order differential equations.

Variation of Parameters

A particular solution $y_p^*(x)$ of the non-homogeneous equation

(1) $$y' + p(x)y = r(x)$$

is given by equation (5) in 2.3, reproduced below in (2). Literature calls it the **variation of parameters** formula or the **variation of constants** formula.

Theorem 4 (Variation of Parameters)
A particular solution of the differential equation $y' + p(x)y = r(x)$ is given by either of the formulas

(2) $$y_p^*(x) = e^{-\int_{x_0}^{x} p(s)ds} \int_{x_0}^{x} r(t)e^{\int_{x_0}^{t} p(s)ds}\,dt,$$

(3) $$y_p(x) = e^{-\int p(x)dx} \int r(x)e^{\int p(x)dx}\,dx.$$

Indefinite Integrals. The indefinite integral form (3) is used in science and engineering applications. The answers (2) and (3) differ by a solution of the homogeneous equation: $y_p^*(x) = y_p(x) + y_h(x)$ for some choice of the constant c in y_h. Both answers (2) and (3) are solutions of the nonhomogeneous differential equation, even though (2) generally contains an extra term. While (2) satisfies $y(x_0) = 0$, (3) may not.

Integrating Factor Formula. An integrating factor for (1) is

$$Q(x) = e^{\int p(x)dx}.$$

Formula (3) can be written in terms of $Q(x)$ as

$$y_p(x) = \frac{1}{Q(x)} \int r(x)Q(x)dx.$$

Compact Formula. Because $\int_x^t f = \int_x^{x_0} f + \int_{x_0}^t f$ and $\int_x^{x_0} f = -\int_{x_0}^x f$, the exponential factors in (2) can be re-written as

(4) $$y_p(x) = \int_{x_0}^{x} r(t)e^{\int_x^t p(s)ds}\,dt.$$

The reader is warned that using indefinite integrals in (4) results in the wrong answer.

Terminology. The name *variation of parameters* comes from the idea of *varying the parameter* c in the homogeneous solution formula $y_h = c\mathbf{R}(x)$, where $\mathbf{R}(x) = e^{-\int p(x)dx}$. Historically, c is replaced by an unknown function $y_0(x)$, to define a *trial solution* $y(x) = y_0(x)\mathbf{R}(x)$ of (1). A derivation appears on page 98.

The Method of Undetermined Coefficients

The method applies to $y' + p(x)y = r(x)$. It finds a particular solution y_p *without* the integration steps present in variation of parameters. The requirements and limitations:

1. Coefficient $p(x)$ of $y' + p(x)y = r(x)$ is constant.
2. The function $r(x)$ is a sum of constants times atoms.

An **atom** is a term having one of the forms

$$x^m,\ x^m e^{ax},\ x^m \cos bx,\ x^m \sin bx,\ x^m e^{ax} \cos bx \quad \text{or} \quad x^m e^{ax} \sin bx.$$

The symbols a and b are real constants, with $b > 0$. Symbol $m \geq 0$ is an integer. The terms x^3, $x \cos 2x$, $\sin x$, e^{-x}, $x^6 e^{-\pi x}$ are atoms. Conversely, if $r(x) = 4 \sin x + 5xe^x$, then split the sum into terms and drop the coefficients 4 and 5 to identify atoms $\sin x$ and xe^x; then $r(x)$ is a sum of constants times atoms.

The Method.

1. Repeatedly differentiate the atoms of $r(x)$ until no new atoms appear. Multiply the distinct atoms so found by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** y.
2. **Correction rule**: if solution e^{-px} of $y' + py = 0$ appears in trial solution y, then replace in y matching atoms e^{-px}, xe^{-px}, ... by xe^{-px}, x^2e^{-px}, ... (other atoms appearing in y are unchanged). The modified expression y is called the **corrected trial solution**.
3. Substitute y into the differential equation $y' + py = r(x)$. Match coefficients of atoms left and right to write out linear algebraic equations for the undetermined coefficients $d_1, \ldots, d_k$.
4. Solve the equations. The trial solution y with evaluated coefficients $d_1, \ldots, d_k$ becomes the particular solution y_p.

Undetermined Coefficients Illustrated. We will solve

$$y' + 2y = xe^x + 2x + 1 + 3 \sin x.$$

Solution:

Test Applicability. The right side $r(x) = xe^x + 2x + 1 + 3\sin x$ is a sum of terms constructed from the atoms xe^x, x, 1, $\sin x$. The left side is $y' + p(x)y$ with $p(x) = 2$, a constant. Therefore, the method of undetermined coefficients applies to find y_p.

Trial Solution. The atoms of $r(x)$ are subjected to differentiation. The distinct atoms so found are 1, x, e^x, xe^x, $\cos x$, $\sin x$ (split terms and drop coefficients to identify new atoms). Because the solution e^{-2x} of $y' + 2y = 0$ does not appear in the list of atoms, then the correction rule does not apply. The corrected trial solution is the expression

$$y = d_1(1) + d_2(x) + d_3(e^x) + d_4(xe^x) + d_5(\cos x) + d_6(\sin x).$$

Equations. To substitute the trial solution y into $y' + 2y$ requires a formula for y':

$$y' = d_2 + d_3 e^x + d_4 x e^x + d_4 e^x - d_5 \sin x + d_6 \cos x.$$

Then

$$\begin{aligned} r(x) &= y' + 2y \\ &= d_2 + d_3 e^x + d_4 x e^x + d_4 e^x - d_5 \sin x + d_6 \cos x \\ &\quad + 2d_1 + 2d_2 x + 2d_3 e^x + 2d_4 x e^x + 2d_5 \cos x + 2d_6 \sin x \\ &= (d_2 + 2d_1)(1) + 2d_2(x) + (3d_3 + d_4)(e^x) + (3d_4)(xe^x) \\ &\quad + (2d_5 + d_6)(\cos x) + (2d_6 - d_5)(\sin x) \end{aligned}$$

Also, $r(x) \equiv 1 + 2x + xe^x + 3\sin x$. Coefficients of atoms on the left and right must match. For instance, constant term $d_2 + 2d_1$ in the expansion of $y' + 2y$ matches constant term 1 in $r(x)$. Writing out the matches gives the equations

$$\begin{aligned} 2d_1 + d_2 &= 1, \\ 2d_2 &= 2, \\ 3d_3 + d_4 &= 0, \\ 3d_4 &= 1, \\ 2d_5 + d_6 &= 0, \\ -d_5 + 2d_6 &= 3. \end{aligned}$$

Solve. The first four equations can be solved by back-substitution to give $d_2 = 1$, $d_1 = 0$, $d_4 = 1/3$, $d_3 = -1/9$. The last two equations are solved by elimination or Cramer's rule (reviewed in Chapter 3) to give $d_6 = 6/5$, $d_5 = -3/5$.

Report y_p. The trial solution y with evaluated coefficients $d_1, \ldots, d_6$ becomes

$$y_p(x) = x - \frac{1}{9}e^x + \frac{1}{3}xe^x - \frac{3}{5}\cos x + \frac{6}{5}\sin x.$$

A Correction Rule Illustration. Solve the equation

$$y' + 3y = 8e^x + 3x^2 e^{-3x}$$