

## Section 1.6 One to one functions, Inverse functions and Logarithms

Recall the definition of a function: "To each  $x$  there is assigned exactly one  $y$ ".

A **function** is called one to one if: "Exactly one  $x$  is assigned to each  $y$ ". (It must be a function first)

Examples:  $f(x) = mx + b$  if  $m \neq 0$ ,  $g(x) = \sqrt{x}$ ,  $h(x) = x^3$  are all one to one.

Example:

One function which is not one to one is  $y = x^2$ . Each non-zero  $y$ -value is assigned to two  $x$ -values.

When a function,  $f$ , is one to one, we can always uniquely answer "If  $f(a)=b$  and we know  $b$ , what is  $a$ ?" because  $b$  was assigned to exactly one  $a$ . For example, if  $f(x) = 2x$  and  $f(x) = 7$  then what is  $x$ ?  $x=7/2$ . If  $g(x)=4x-5$  and  $g(x)=19$  then  $x=6$ .

For the examples:

$$f(x) = y = mx + b \quad x = \frac{y - b}{m}$$

$$g(x) = y = \sqrt{x} \quad x = y^2$$

$$h(x) = y = x^3 \quad x = \sqrt[3]{y}$$

Inverse functions: In the above examples, if we switch places with  $x$  and  $y$ , we have the inverse functions.

	Inverse function
$f(x) = y = mx + b \quad x = \frac{y - b}{m}$	$y = \frac{x - b}{m}$
$g(x) = y = \sqrt{x} \quad x = y^2$	$y = x^2$
$h(x) = y = x^3 \quad x = \sqrt[3]{y}$	$y = \sqrt[3]{x}$

The notation for the inverse function to  $f(x)$  is  $f^{-1}(x)$ .

This is similar to the notation for multiplicative inverses but should not be confused.

Ex. If  $f(x) = 2x$  then the inverse function is  $f^{-1}(x) = \frac{1}{2}x = 2^{-1}x$ .

This is not the same as  $(f(x))^{-1} = (2x)^{-1} = \frac{1}{2x}$ .

When the exponent  $-1$  is on a number, it means the multiplicative inverse. When it is on the name of the function, such as  $f$ , it means the inverse function.

Ex.  $h(x) = x^3 \quad h(2) = 8 \quad h^{-1}(8) = 2 \quad (h(8))^{-1} = (8^3)^{-1} = (512)^{-1} = \frac{1}{512}$

An important fact to remember:  $f(f^{-1}(x))=x$  and  $f^{-1}(f(x)) = x$

Ex. For  $h(x) = x^3$   $h^{-1}(x) = \sqrt[3]{x} = (x)^{1/3}$ ;

$$h(h^{-1}(x)) = (\sqrt[3]{x})^3 = x$$

$$h^{-1}(h(x)) = \sqrt[3]{x^3} = x$$

The graph of  $f^{-1}(x)$  is a reflection across the line  $y = x$  of the graph of  $f$ .

You can see this in your calculator by graphing  $Y1 = f(x)$ ,  $Y2 = x$  and  $Y3 = f^{-1}(x)$  for each of the examples above.

The domain of  $f$  is the range of  $f^{-1}$  and the range of  $f$  is the domain of  $f^{-1}$ .

**Logarithms:** For any positive  $b$ ,  $f(x) = b^x$  is a one to one function. So it has an inverse function and the inverse is called  $\log_b x$ , the logarithm base  $b$  of  $x$ .

When  $b=10$ , we write  $\log x$ . Whenever  $b$  is not given in the logarithm, assume  $b=10$ .

When  $b=e$ , we write  $\ln x$ .  $\ln x$  is called the natural logarithm.

**The logarithm is the exponent** we raise  $b$  to so that power of  $b$  will be  $x$ .

The domain of  $\log_b x$  is  $(0, \infty)$ . That is,  $x$  must be positive.

Examples:

$\log 100=2$ , because  $10^2 = 100$ . (The base is not given so it is 10 and the 2nd power of 10 is 100.)

$$\log_2 4 = 2 \quad \text{because} \quad 2^2 = 4 \qquad \log_3 9 = 2 \quad \text{because} \quad 3^2 = 9$$

$$\log_2 8 = 3 \qquad 2^3 = 8 \qquad \log_3 27 = 3 \qquad 3^3 = 27$$

$$\log_2 16 = 4 \qquad 2^4 = 16 \qquad \log_3 81 = 4 \qquad 3^4 = 81$$

Since exponential grow rapidly for  $b>1$ , sometimes logarithmic scales are used.

Example: The richter scale which measure earthquakes is log base 10 scale.

If the value on the richter scale increases by one then the magnitude of the earthquake is 10 times as great. An earthquake which measures 6 on the richter scale is 100 times as strong as one that measures 4.

The pH of a solution is  $-\log H^+$  where  $H^+$  is the hydrogen ion concentration.

If the pH increases by 1, then  $H^+$  is 1/10th as much.

**Laws of Logarithms:**

These follow from exponent laws.

$$\log_b (MN) = \log_b M + \log_b N \qquad \log_b \frac{M}{N} = \log_b M - \log_b N \qquad \log_b (M^r) = r \log_b M$$

Note:

There is no rule for  $(\log_b M)^r$ . Use parentheses where needed to avoid confusion.

The logarithm of an exponential function is linear:  $\ln(ab^x) = \ln a + x \ln b$

Use logarithms when you need to solve for an exponent.

Example: If any amount of money is invested at 7% annual interest compounded continuously, how long will it take to double the original amount?

$A(t) = Pe^{.07t}$  To double this we solve  $Pe^{.07t} = 2P$  so  $e^{.07t} = 2$

Since the unknown is in the exponent, we take  $\ln$  of both sides to get

$$\ln e^{.07t} = \ln 2 \quad .07t = \ln 2 \quad t = \frac{\ln 2}{.07} \approx 10 \text{ years}$$

In general, the doubling time for continuous compound interest is  $\frac{\ln 2}{r}$  where  $r$  is the interest rate as a decimal.

Example: Write as a single logarithm and evaluate without a calculator,  
 $3 \log_2 48 - \log_2 (6^3)$ . First bring down the exponent of 3 from the 6. Then

$$3 \log_2 48 - 3 \log_2 6 = 3[\log_2 48 - \log_2 6] = 3 \log_2 \frac{48}{6} = 3 \log_2 8 = 9$$

Solve for  $x$ : a)  $\log_2 (x^2 - 1) = 3$

Solution: 3 is the exponent of 2 so that  $x^2 - 1 = 2^3 = 8$   $x^2 = 9$   $x = 3$  or  $x = -3$

b)  $\log_2 (x + 1) + \log_2 (x - 1) = 3$  Using the sum of logs rule, we have

$\log_2 (x^2 - 1) = 3$ . So we know  $x=3$  or  $x=-3$  but now  $x=-3$  cannot be substituted in the original expressions so  $x=3$  is the only solution.