

Section 2.2 Limits

In everyday speech, we often talk about spending limits. We are talking about the most you can spend. You need not get close to your spending limit, you just cannot go over it. Mathematically, this is not a limit but is an upper bound.

Consider the sequence of numbers:

$1/2, -1/2, 1/3, -1/3, 1/4, -1/4, 1/5, -1/5, 1/6, -1/6 \dots$ The magnitude of all the numbers gets smaller and smaller the farther out we go. The limit of this sequence is 0 because we can get all the numbers as close as we want to 0 by going far enough along the sequence. Notice that 0 is neither an upper or a lower bound for the numbers. 0 is the number squeezed in on.

Example. What is the limit of the sequence $.3, .33, .333, .3333, .33333, \dots$ The farther out we go, the closer all the numbers get to $1/3$ so $1/3$ is the limit. It also is an upper bound but that is not the point.

What is the limit of a function, $f(x)$, as x approaches a ?

1) Do not let x actually equal a .

2) There is a number L so that all values of $f(x)$ are as close as desired to L if x is close enough to a .

This means we can put smaller and smaller rectangles (widths and lengths approaching 0) around (a, L) so that the graph stays inside the rectangles except possibly at $(a, f(a))$.

The y -values squeeze in on the number L if x squeezes in on a but doesn't equal a .

We write $\lim_{x \rightarrow a} f(x) = L$ or $f(x) \rightarrow L$ as $x \rightarrow a$.

Example: $\lim_{x \rightarrow 3} x^2 = 9$, $\lim_{x \rightarrow 1} (x^2 + 2x + 5) = 8$

In fact if the function is a polynomial, $p(x)$, then $\lim_{x \rightarrow a} p(x) = p(a)$. You just plug a into the

rule for the polynomial and that is the limit. The same works for rational functions as long as the denominator is not 0 at a . The problems come when you cannot substitute a into the rule and get a meaningful answer, or when the function is piecewise.

Example: Find $\lim_{x \rightarrow 2} f(x)$ for $f(x) = \begin{cases} x^2 & x < 2 \\ 6 - x & x > 2 \end{cases}$. Notice $f(2)$ is not defined but that does

not affect the limit as x approaches 2.

Graph this function and see that the two pieces meet. $\lim_{x \rightarrow 2} x^2 = 4$ and $\lim_{x \rightarrow 2} (6 - x) = 4$.

Since both sides approach the same number, 4, all the y -values approach 4 as x approaches 2. The limit is 4.

Example: Find $\lim_{x \rightarrow 0} x \sin(1/x)$. Graph $x \sin(1/x)$ and you will see the graph wobbles up and down but approaches $(0, 0)$ near $x=0$. We cannot substitute $x=0$ into this function so we had to use another method. You could say that $\sin(1/x)$ is always between -1 and 1 so $x \sin(1/x)$ is always between $-x$ and x and so it goes to 0 as x goes to 0 .

Right and Left hand limits. Sometimes the graph for $x < a$ and the graph for $x > a$ approach different y -values. This happens often in piecewise functions.

The left hand limit is $\lim_{x \rightarrow a^-} f(x)$. The upper minus on a means x stays less than a .

The right hand limit is $\lim_{x \rightarrow a^+} f(x)$. The upper plus on a means x stays greater than a .

Example: $f(x) = \begin{cases} x^2 + 5x & x < 3 \\ -x^2 & x > 3 \end{cases}$ $\lim_{x \rightarrow 3^-} f(x) = 24$ $\lim_{x \rightarrow 3^+} f(x) = -9$

When the right and left hand limits are different, the unsigned limit does not exist.

$\lim_{x \rightarrow 3} f(x)$ DNE (does not exist).

Another way a limit can fail to exist is when there is a vertical asymptote.

Example: $\lim_{x \rightarrow 0} \frac{1}{x}$ DNE. One side goes down to minus infinity and the other goes up to plus infinity.

Still another way a limit can fail to exist is if the function alternates infinitely often or oscillates between two or more numbers.

Example: Graph $f(x) = \sin(1/x)$ with $X_{\min} = -.5$ and $X_{\max} = .5$, $Y_{\min} = -1$, $Y_{\max} = 1$.

This function takes on all values between -1 and 1 in every x -interval containing 0 no matter how small we make the interval.