Section 2.4 Continuity

Definition: A function, *f(x)*, is **continuous at *a*** if and only if $\lim_{x \to a} f(x) = f(a)$. This statement implies that *f(a)* is defined, $\lim_{x \to a} f(x)$ exists as a number, and the two are equal.

A function is **continuous on an interval** if it is continuous at every *a* in the interval.

The graph of a continuous function has no holes, gaps, or vertical asymptotes. You can draw the graph without lifting the pencil.

Polynomials, sin*x* and cos*x*, and exponential functions are continuous on the whole real line.

Rational functions and the trigonometric functions $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$ are continuous wherever the denominator is nonzero and are not continuous (discontinuous) wherever the denominator is 0.

Left and right hand continuity: A function, *f(x)* is continuous from the left at *a* if and only if $\lim_{x \to a^-} f(x) = f(a)$. Similarly, *f(x)* is continuous from the right at *a* if and only if $\lim_{x \to a^+} f(x) = f(a)$.

Example: $f(x) = \sqrt{x}$ is continuous from the right at 0. We say *f* is continuous on $[0, \infty)$.

Examples: $f(x) = \frac{x^2 - 2x + 5}{x^2 - 9}$ is continuous except at 3 and -3. *f* has vertical asymptotes x = 3

and x = -3.

$g(x) = \frac{x^2 - 2x - 3}{x^2 - 9}$ is also continuous except at 3 and -3. *g* has a hole at x=3 and

vertical asymptote

x=-3. **What do we mean by a hole?** The limit as *x* approaches a exists but is not equal to *g(a)*.

To see this, factor the numerator and the denominator :

$$\frac{x^2 - 2x - 3}{x^2 - 9} = \frac{(x-3)(x+1)}{(x-3)(x+3)} = \frac{x+1}{x+3} \quad \text{if} \quad x \neq 3$$

$\lim_{x \to 3} g(x) = \frac{3+1}{3+3} = \frac{2}{3}$ This type of discontinuity is called **removable** since we only need to define

g(3)=$\frac{2}{3}$ to make it continuous at 3. The discontinuity at -3 is non-removable.

What is a gap? A piecewise function has a gap at *a* if the left piece does not meet the right piece at *a*.

This means $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. Since the limit does not exist at *a*, the function is not continuous at *a*.

Example: $f(x) = \begin{cases} 3x+4 & x \leq 1 \\ x^2 & 1 < x < 4 \\ 4x & x \geq 4 \end{cases}$ $\lim_{x\to 1^-} f(x) = 3 + 4 = 7$, $\lim_{x\to 1^+} f(x) = 1^2 = 1$ shows *f* is not continuous at 1. The graph has a gap there.

(Since 7 is the value of *f* and is the left hand limit, *f* is left continuous at 1, but it is not right continuous.)

Check x=4: $\lim_{x\to 4^-} f(x) = 4^2 = 16$ $\lim_{x\to 4^+} f(x) = 4\cdot 4 = 16$ and *f*(4)=16 so *f* is continuous at 4.

Intermediate Value Theorem: If *a*<*b* and *f* is continuous on the interval [*a, b*], and $f(a) \neq f(b)$ then *f* assumes every value between *f(a)* and *f(b)* on the interval [a, b].

The reason is:

We cannot draw a continuous curve from (a, f(a)) to (b, f(b)) without crossing every horizontal line y=C for C between *f(a)* and *f(b)*.

How do we use the Intermediate Value Theorem?

Example: Show that the polynomial $p(x) = x^3 + 5x - 3$ has a root in the interval [-1, 1].

p(1)= 3 p(-1) = -9 so there is some *c* between -1 and 1 where *p(c)*=0.