Section 2.7 The Derivative as a Function

Differentiable: A function, f(x), is differentiable at x=a means f '(a) exists. If the derivative exists on an interval, that is , if f is differentiable at every point in the interval, then the derivative is a function on that interval.

Definition: $f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$.

Example:

$$f(x) = x^2 \qquad f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} (2x + h) = 2x$$

We also know the derivative of a line is the slope of the line. Applying the linear rule for derivatives, we can differentiate any quadratic.

Example: $g(x) = 4x^2 - 6x + 7 \qquad g'(x) = 4(2x) - 6 = 8x - 6$

We can find the tangent line to *g* at a given point.

Example: For *g(x)* as above, find the equation of the tangent line at (2, *g*(2)).

g(2)=4(4)-12+7= 11 so the point of tangency is (2, 11).

The slope is *g*'(2)= 8(2)-6=10 so the tangent line is
y = 10(*x* - 2) + 11

Notations for the derivative:

$$f'(x) = \frac{df}{dx} \qquad f'(a) = \left. \frac{df}{dx} \right|_{x=a}$$

If a function is differentiable at x=a then it must be continuous at a.
Contrapositive:
If a function is not continuous at x=a then it is not differentiable at a.
The following example illustrates why.

Example: $f(x) = \begin{cases} x^2 & x \neq 2 \\ 5 & x = 2 \end{cases}$ This is not continuous at 2 so why does that mean it is not differentiable at 2?

$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 - 5}{h}$ As h approaches 0, the numerator approaches -1 and the denominator approaches 0, the left side approaches infinity and the right side approaches minus infinity and the limit does not exist. Graphically, you cannot draw a line tangent to the graph at x=2 and passing through (2, 5).

A function is not differentiable where it has a corner, a cusp, a vertical tangent, or at any discontinuity. These are some possibilities we will cover.

Examples of corners and cusps.

1. $f(x) = |x^2 - 4|$ This function turns sharply at -2 and at 2. It is not differentiable at x= -2 or at x=2. To graph it, sketch the graph of $x^2 - 4$ and reflect the region where y is negative across the x-axis.

2. $f(x) = x^{2/3}$ Graph it in your calculator and you will see the cusp at (0, 0).

3. When a piecewise function is continuous at a but the left and right pieces meet at different slopes, the function has a corner at a.

Example: $f(x) = \begin{cases} x^2 & x < 3 \\ 5x - 6 & 3 \leq x \end{cases}$ Check that *f* is continuous at 3.

The slope of the left piece at 3 is the derivative of x^2 at x=3. We found the derivative of x^2 is 2*x* so the slope of the left piece at 3 is 6.
The slope of the right piece is 5. The pieces meet at an obtuse corner.

A more obvious corner occurs in |x| at x=0.

Vertical Tangents occur when *f* is continuous but *f'* has a vertical asymptote.
$g(x) = x^{1/3}$ has a vertical tangent at x=0.
Example: $f(x) = (x^2 - 4)^{1/3}$ Graph this function in your calculator. It has vertical tangents at x= - 2 and at x=2. We will learn how to find this derivative in later sections.

Higher Order Derivatives: Since *f'* is a function, we can try to find its derivative.
The derivative of *f'* is the 2nd derivative of f and written

$f''(x) = \frac{d^2 f}{dx^2}$.

Example: $f(x) = x^2 \quad f'(x) = 2x \quad f''(x) = 2$

Velocity is the derivative of distance traveled and acceleration is the derivative of velocity, so acceleration is the 2nd derivative of distance.

Example: The height of an object shot straight upward from 60 ft above the ground with initial velocity 50 ft per sec has height given by $h(t) = 50 + 60t - 16t^2$. The velocity at t sec. is $h'(t) = v(t) = 60 - 32t$ ft/sec and the acceleration is $h''(t) = v'(t) = -32$ ft/sec^2 which is the acceleration downward due to gravity.