Perfect Bayesian Equilibrium in Sender-Receiver Games

Introduction

We previously studied static games of imperfect information.[1] Each player $i \in I$ has private information which was summarized by her type $\theta_i \in \Theta_i$. Each player knows her own type but does not in general know the types of her opponents. Each player i's beliefs about the types $\theta_{-i} \in \Theta_{-i}$ of her opponents are derived from her knowledge of her own type θ_i and a common prior belief $p \in \Delta(\Theta)$ over the space of type profiles.

Nature moves first, picking a type profile $\theta \in \Theta$ according to the probability distribution $p \in \Delta(\Theta)$. Nature then privately informed each player $i \in I$ of her type: player 1 is type θ_1, player 2 is type $\theta_2, \ldots,$ player n is type θ_n. Then each player $i \in I$ of the n players simultaneously chooses an action $a_i \in A_i$ from her action space. A payoff $u_i(a, \theta)$ is then awarded to each player, which depends on the action profile $a \in A$ the players chose and the type profile θ Nature chose.

It was because the players simultaneously chose their actions that we called these games static. Now we want to generalize our analysis by considering dynamic games of incomplete information; i.e. we consider games in which some players take actions before others and these actions are observed to some extent by some other players.

Sender-receiver games

We consider here the simplest dynamic games of incomplete information: *sender-receiver* games. There

1 See the "Static Games of Incomplete Information" handout.

are only two players: a Sender (S) and a Receiver (R). The Sender's action will be to send a *message* $m \in M$ chosen from a *message space* M to the Receiver. The Receiver will observe this message m and respond to it by choosing an action $a \in A$ from his action space A.

To make this game a simple but nontrivial game of incomplete information we endow the Sender with some private information which we describe by her type $\theta \in \Theta$. The Receiver has no private information, so he has but a single type, which we then have no need to mention further. The Receiver does have prior beliefs (i.e. prior to observing the Sender's message) about the Sender's type, which are described by the probability distribution $p \in \Delta(\Theta)$ over the Sender's type space Θ. In other words, before observing the Sender's message, the Receiver believes that the probability that the Sender is some particular type $\theta \in \Theta$ is $p(\theta)$.

We will typically assume that the type space Θ, the message space M, and the Receiver's action space A are finite sets.

After the Receiver takes an action $a \in A$, each player is awarded a payoff which can in general depend on the message m the Sender sent, the action a the Receiver took in response, and the type θ which Nature chose for the Sender. The payoffs to the Sender and Receiver to a (message, action, type) triple $(m, a, \theta) \in M \times A \times \Theta$ are $u(m, a, \theta)$ and $v(m, a, \theta)$, respectively. I.e. $u, v: M \times A \times \Theta \to \mathbb{R}$.

We can express this game of incomplete information as an extensive-form game of imperfect information by explicitly representing Nature, who chooses a type $\theta \in \Theta$ for the Sender. Because the Sender observes this choice of Nature, every Sender information set is a singleton, and the number of Sender information sets is equal to the number of possible Sender types, viz. $\#\Theta$. The Receiver observes only the message sent by the Sender. Therefore the number of Receiver information sets is equal to the number of possible messages the Sender can transmit, viz. $\#M$. Within each of his information sets the Receiver cannot distinguish between the Sender's possible types; so each Receiver information set has a number of nodes equal to the number of possible Sender types, viz. $\#\Theta$. Therefore the total number of Receiver nodes is the product of the cardinalities of the message and type spaces, viz. $\#M \cdot \#\Theta$.

Example: Life's a Beach?

For example, the Sender might be a job applicant and the Receiver an employer. The Sender's decision could be a choice between going to College or going to the Beach. The particular choice the Sender makes is her message.[2] The employer will observe this decision and decide to Hire or Reject the applicant. In this example the Sender's message space is $M = \{\text{College, Beach}\}$ and the Receiver's action space is $A = \{\text{Hire, Reject}\}$.

The Sender's private information could concern her aptitude: she knows whether she is Bright or

[2] Of course in this case the message seems more than a mere message. The terms "message" for the Sender and "action" for the Receiver both refer to actions taken by a player. The distinction between the two is only interpretational. We use "message" for the Sender's action to acknowledge that the Sender realizes that the Receiver will respond to the Sender's action, and therefore the Sender can attempt to influence the Receiver's response through her choice of message.

Dull. Therefore her type space would be $\Theta = \{\text{Bright}, \text{Dull}\}$. The Receiver's prior beliefs concerning the probability that the Sender is Bright or Dull can be described by a single number $\gamma \in [0, 1]$. With probability γ, the Sender is Bright; with probability $1-\gamma$, she is Dull. I.e. the Receiver's prior beliefs $p \in \Delta(\Theta)$ are defined by $p(\text{Bright}) = \gamma$ and $p(\text{Dull}) = 1-\gamma$.

Consider the simple sender-receiver game shown in Figure 1. Note that the Sender has two information sets, corresponding to her two types (viz. Bright and Dull). The Receiver also has two information sets, but these correspond to the Sender's two possible messages (viz. Beach and College) rather than to the Sender's possible types. (The Receiver's left-hand information set is his Beach information set and his right-hand information set is his College information set.)

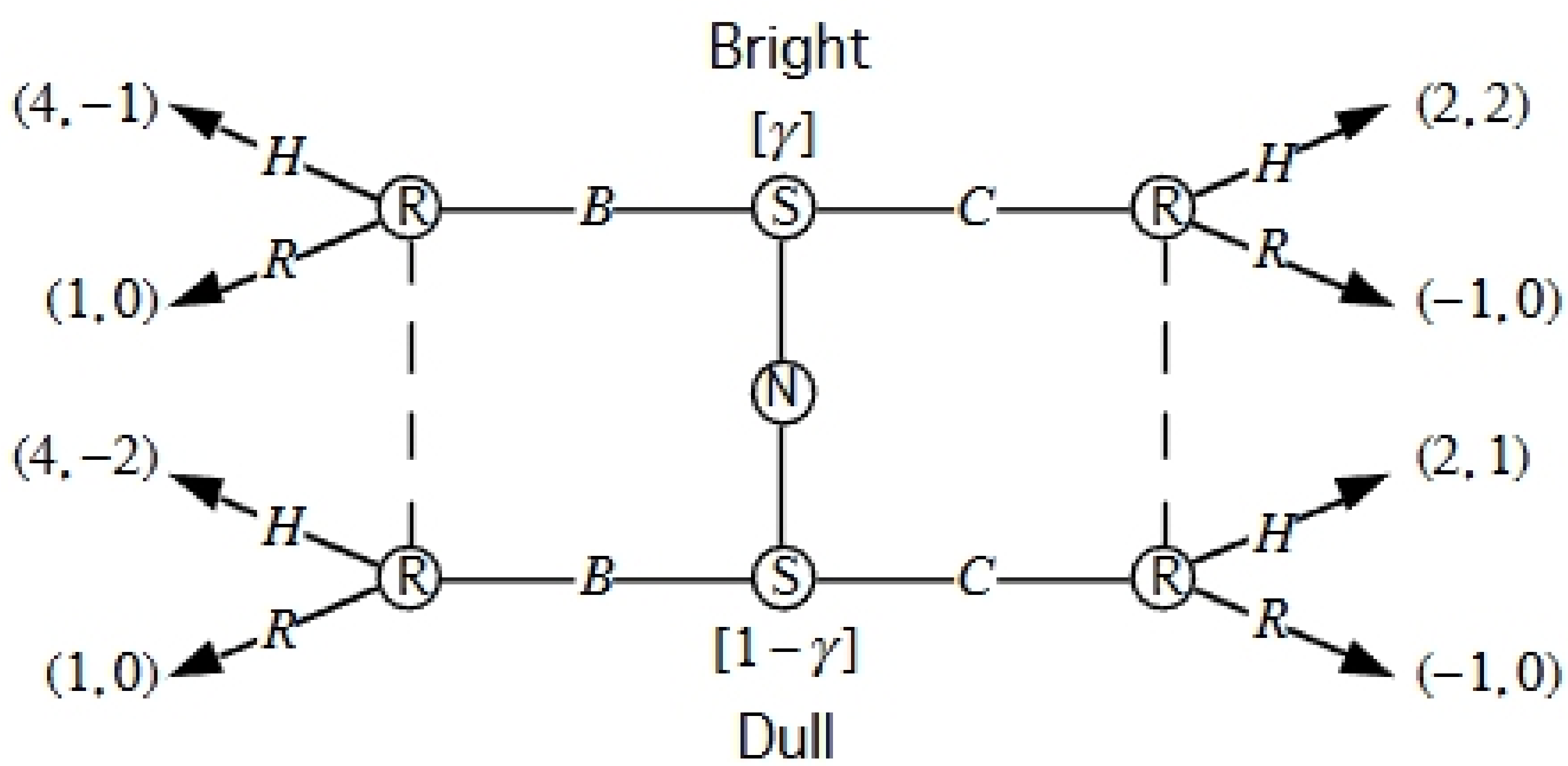

Figure 1: A simple sender-receiver game.

Let's interpret the payoffs shown in Figure 1. The first and second payoffs of each ordered pair are the Sender's and Receiver's payoffs, respectively, for a particular type/message/action triple. For a fixed type and Receiver action, the Sender's payoff to going to the Beach is always two greater than her payoff to going to College.[3] For fixed educational and employment decisions, the Sender's payoff is independent of her type.[4] For a fixed type and educational decision, the Sender receives a payoff from being Hired which is 3 greater than her payoff if she is rejected.[5] To summarize the Sender's payoffs (with the appropriate *ceteris paribus* qualifications): the Sender prefers the Beach over going to College, prefers being Hired over being Rejected, and is not discriminated against due to aptitude.

Whenever the Receiver Rejects an applicant, the Receiver gets a payoff of zero. Although the Sender's aptitude did not directly influence the Sender's payoffs, aptitude is payoff-relevant to the Receiver when he Hires: For a fixed educational decision, the Receiver's payoff to Hiring is 1 greater

3 For example, If the Bright applicant is Hired, she receives a payoff of 4 from the Beach but only 2 from College. If the Dull applicant is Rejected, she receives a payoff of 1 from the Beach but only −1 from College.

4 For example, if the applicant goes to the Beach and is Hired, she receives a payoff of 4 regardless of whether she is Bright or Dull.

5 For example, if the Bright applicant goes to College, she receives 2 if she is Hired and only −1 if she is Rejected. If the Dull applicant goes to the Beach, she receives 4 if she is Hired and only 1 if she is Rejected.