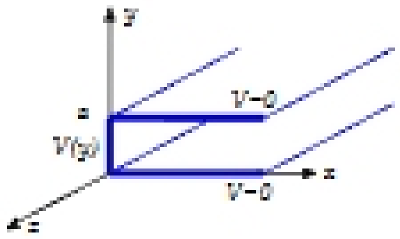
Today in Physics 217: separation of variables

- Introduction to the method, in Cartesian coordinates.
- Example solution for the potential in an infinite slot, arbitrary V at the bottom, in which we introduce two common features of separation solutions:
  - Completeness and orthogonality of sines
  - Fourier's trick



Introduction to separation of variables

- If the method of images can't be used on a problem you must solve that involves conductors, the next thing to try is direct solution of the Laplace equation, subject to boundary conditions on V and/or $\partial V/\partial n$.
- Separation of variables is the easiest direct solution technique. It works best with conductors for which the surfaces are well behaved (planes, spheres, cylinders, *etc.*).

Here's how it works, in Cartesian coordinates, in which the Laplace equation is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

which can't be integrated directly like the 1-D case.

Introduction to separation of variables (continued)

Consider solutions of the form

$$V(x,y,z) = X(x)Y(y)Z(z)$$

Then, the Laplace equation is

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

or, dividing through by XYZ,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

i.e. $$f(x) + g(y) + h(z) = 0$$

Introduction to separation of variables (continued)

The only way for this to be true for all x,y,z is for each term to be a constant, and for the three constants to add up to zero:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=A+B-(A+B)=0$$

So this is a way to trade one partial differential equation for three ordinary differential equations, which can often be solved much more easily:

$$\frac{d^2X}{dx^2}-AX=0 \qquad \frac{d^2Y}{dy^2}-BY=0 \qquad \frac{d^2Z}{dz^2}+(A+B)Z=0$$

Introduction to separation of variables (continued)

Nothing guarantees that V will always factor into functions of x, y, and z alone. In fact, there are certainly many solutions to the Laplace equation that are not of this form. However,

- ❑ there are plenty of electrostatic problems for which the boundary conditions are specified on well-behaved surfaces, and do turn out to have solutions of this form, and
- ❑ the solutions to electrostatics problems are unique, so if separation of variables yields a solution at all, it's guaranteed to be the correct one.

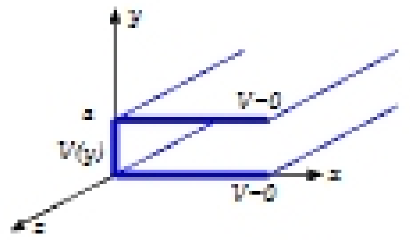
Separation of variables is also a very useful PDE solution technique in quantum mechanics, where one finds many problems in which the boundary conditions are specified on regular, well-behaved surfaces.

Example: the infinite slot

Griffiths, example 3.3: Two infinite, grounded, metal plates lie parallel to the x-z plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates and maintained at a specified potential $V_0(y)$. Find the potential inside this slot.

No z dependence, so

$$V=V(x,y)$$



Semi-infinite conducting plates: $z=-\infty\to\infty,\ x=0\to\infty,\ y=0,a.$ Bottom of the slot: plate insulated from the other two, with $V=V_0(y)$.

The infinite slot (continued)

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Suppose $V(x,y) = X(x)Y(y)$; then

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \quad \Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 = k^2 - k^2$$

or:

$$\text{I.} \quad \frac{d^2X}{dx^2} - k^2X = 0 \qquad \text{II.} \quad \frac{d^2Y}{dy^2} + k^2Y = 0$$

We chose k^2 rather than, say, A, to indicate that this constant is non-negative, and that the other one $(-k^2)$ is non-positive. Why? We'll see in a minute.

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The infinite slot (continued)

Boundary conditions:

1. $V \to 0$ as $x \to \infty$ (reference point at infinity)
2. $V = 0$ at $y = 0$
3. $V = 0$ at $y = a$
4. $V = V_0(y)$ at $x = 0$

Solutions: It turns out that you know equations I and II very well from MTH 165. But the means by which they're solved is too useful to forget, so I'll remind you, first, with I.

$$\frac{d^2X}{dx^2} = k^2X$$

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The infinite slot (continued)

Let $v = dX/dx$, and multiply through by $v = dX/dx$:

$$v\frac{dv}{dx} = k^2X\frac{dX}{dx}$$

Integrate over x: $\int v\frac{dv}{dx}dx = k^2\int X\frac{dX}{dx}dx$

$$\frac{v^2}{2} = k^2\frac{X^2}{2} + S$$

Here both X and dX/dx must be zero for all x at $y = 0$ and a, so the constant $S = 0$:

$$\frac{dX}{dx} = kX \text{ or } -kX$$

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