Recursive definitions of sets

- We saw last time how to define the set of propositional expressions recursively.
- The method extends to other kinds of sets: Let S be the subset of $\mathbb{N}$ defined by the following rules:
  1. $3 \in S$;
  2. if $x \in S$ and $y \in S$ then $x + y \in S$;
  3. No number is in S unless it can be shown to be there using (1) and (2).
- Example:
  - $3 \in S$ (rule 1)
  - $6 = 3 + 3 \in S$ by 1. and rule 2
  - $9 = 6 + 3 \in S$ by 2. and rule 2.

Using induction along with recursive definitions

Proposition *The set S on the last slide is the set of positive multiples of 3.*

Proof: Let P be the set of positive multiples of 3. We show $P \subseteq S$ and $S \subseteq P$. For the first inclusion, we show that every integer of the form $3n$, for $n \geq 1$, is in S. We do this by induction on n.

Basis. When $n = 1$, the number $3 \cdot 1 \in S$ by rule 1.

Induction step. Assume that $3k$ is in S. We want $3(k+1) \in S$. But $3(k+1) = 3k + 3$. By inductive hypothesis $3k \in S$, and 3 is already in S, so $3k + 3 \in S$ by rule 2.

Proof continued: $S \subseteq P$.

To show this, we rely on rule 3, which says nothing is in S unless you can show it in a finite number of uses of rules 1 and 2. Let n be the number of uses of these rules. We show by induction on n that the integer proved to be in S is in fact a positive multiple of 3.

Basis. We just apply one rule, which has to be rule 1. This rule shows $3 \in S$, and 3 is a positive multiple of 3.

Induction step (strong form). Assume that whenever we show that $p \in S$ by using k or fewer steps, then p is a positive multiple of 3. Consider a "proof" using $k+1$ steps. The last rule used in this proof is rule 2, which says that if x and y are in S, so is $x+y$. Now x was shown in S by $\leq k$ steps, and so was y. By inductive hypothesis (twice), we know that x and y are positive multiples of 3, and therefore so is $x+y$.