

Sets

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Introduction to Discrete Mathematics
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Introduction I

We've already implicitly dealt with sets (integers, \mathbb{Z} ; rationals (\mathbb{Q}) etc.) but here we will develop more fully the definitions, properties and operations of sets.

Definition

A set is an unordered collection of (unique) objects.

Sets are fundamental discrete structures that form the basis of more complex discrete structures like graphs.

Contrast this definition with the one in the book (compare *bag*, *multi-set*, *tuples*, etc).

Definition

Introduction II

The objects in a set are called *elements* or *members* of a set. A set is said to *contain* its elements.

Recall the notation: for a set A , an element x we write

$$x \in A$$

if A contains x and

$$x \notin A$$

otherwise.

Latex notation: `\in`, `\notin`.

Terminology I

Definition

Two sets, A and B are equal if they contain the same elements. In this case we write $A = B$.

Example

$\{2, 3, 5, 7\} = \{3, 2, 7, 5\}$ since a set is *unordered*.

Also, $\{2, 3, 5, 7\} = \{2, 2, 3, 3, 5, 7\}$ since a set contains *unique* elements.

However, $\{2, 3, 5, 7\} \neq \{2, 3\}$.

Terminology II

A *multi-set* is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \dots, m_r \cdot a_r\}$ is a set where m_1 occurs a_1 times, m_2 occurs a_2 times, etc.

Note in CS (Databases), we distinguish:

- ▶ a set is w/o repetition
- ▶ a bag is a set with repetition

Terminology III

We've already seen *set builder* notation:

$$O = \{x \mid (x \in \mathbb{Z}) \wedge (x = 2k \text{ for some } k \in \mathbb{Z})\}$$

should be read O is the set that contains all x such that x is an integer and x is even.

A set is defined in *intension*, when you give its set builder notation.

$$O = \{x \mid (x \in \mathbb{Z}) \wedge (x \leq 8)\}$$

A set is defined in *extension*, when you enumerate all the elements.

$$O = \{0, 2, 4, 6, 8\}$$

Venn Diagram

Example

A set can also be represented graphically using a *Venn diagram*.

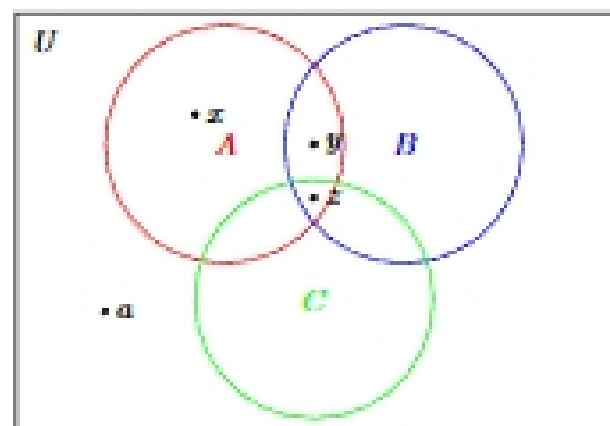


Figure: Venn Diagram

More Terminology & Notation I

A set that has no elements is referred to as the *empty set* or *null set* and is denoted \emptyset .

A *singleton set* is a set that has only one element. We usually write $\{a\}$. Note the different: brackets indicate that the object is a set while a without brackets is an *element*.

The subtle difference also exists with the empty set: that is

$$\emptyset \neq \{\emptyset\}$$

The first is a set, the second is a set containing a set.

More Terminology & Notation II

Definition

A is said to be a *subset* of B and we write

$$A \subset B$$

if and only if every element of A is also an element of B .

That is, we have an equivalence:

$$A \subset B \iff \forall x(x \in A \rightarrow x \in B)$$

More Terminology & Notation III

Theorem

For any set S ,

- ▶ $\emptyset \subset S$ and
- ▶ $S \subset S$

(Theorem 1, page 81.)

The proof is in the book—note that it is an excellent example of a *vacuous proof*!

Latex notation: `\emptyset`, `\subset`, `\subseteq`.

More Terminology & Notation IV

Definition

A set A that is a subset of B is called a *proper subset* if $A \neq B$.

That is, there is some element $x \in B$ such that $x \notin A$. In this case we write $A \subset B$ or to be even more definite we write

$$A \subsetneq B$$

Example

Let $A = \{2\}$. Let $B = \{x \mid (x \leq 100) \wedge (x \text{ is prime})\}$. Then $A \subsetneq B$.

Latex notation: `\subsetneq`.

More Terminology & Notation V

Sets can be elements of other sets.

Example

$$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

and

$$\{\{1\}, \{2\}, \{3\}\}$$

are sets with sets for elements.

More Terminology & Notation VI

Definition

If there are exactly n distinct elements in a set S , with n a nonnegative integer, we say that S is a finite set and the cardinality of S is n . Notationally, we write

$$|S| = n$$

Definition

A set that is not finite is said to be *infinite*.

More Terminology & Notation VII

Example

Recall the set $B = \{x \mid (x < 100) \wedge (x \text{ is prime})\}$, its cardinality is

$$|B| = 25$$

since there are 25 primes less than 100. Note the cardinality of the empty set:

$$|\emptyset| = 0$$

The sets \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} are all infinite.

Proving Equivalence I

You may be asked to show that a set is a subset, proper subset or equal to another set. To do this, use the equivalence discussed before:

$$A \subseteq B \iff \forall x(x \in A \rightarrow x \in B)$$

To show that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x , $x \in A$ implies that x is also in B . Any proof method could be used.

To show that $A \subsetneq B$ you must show that A is a subset of B just as before. But you must also show that

$$\exists x((x \in B) \wedge (x \notin A))$$

Finally, to show two sets equal, it is enough to show (much like an equivalence) that $A \subseteq B$ and $B \subseteq A$ independently.

Proving Equivalence II

Logically speaking this is showing the following quantified statements:

$$(\forall x(x \in A \rightarrow x \in B)) \wedge (\forall x(x \in B \rightarrow x \in A))$$

We'll see an example later.

The Power Set I

Definition

The power set of a set S , denoted $\mathcal{P}(S)$ is the set of all subsets of S .

Example

Let $A = \{a, b, c\}$ then the power set is

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Note that the empty set and the set itself are always elements of the power set. This follows from Theorem 1 (Rosen, p81).

The Power Set II

The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set.

Fact

Let S be a set such that $|S| = n$, then

$$|\mathcal{P}(S)| = 2^n$$