

# ME 201/MTH 281/ME 400/CHE 400

## ASSIGNMENT #7 2011

Assignments handed in by 6 PM on Wednesday Oct. 26 will receive a 5-point bonus. Assignments handed in after that but by 6 PM on Thursday Oct. 27 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday.

### LECTURE SCHEDULE AND READING

| <u>Section in Class Notes</u>                    | <u>Date</u> | <u>Section in Text</u> |
|--|-------------|------------------------|
| V. SEPARATION OF VARIABLES, PART 2               |             |                        |
| 5.1 Heat Conduction with Newton's Law of Cooling | Th Oct 20   | 5.8                    |
| 5.2 Heat Conduction with Sources                 | F Oct 21    | 8.3                    |
| 5.3 Critical Size of a Nuclear Reactor           | M Oct 24    | ---                    |

### PROBLEMS

(1) (40 points)

(a) (20 points) Find the first three eigenvalues and eigenfunctions of the Sturm-Liouville problem given below.

$$\frac{d^2 y}{dx^2} + \lambda y = 0, \quad 0 < x < 1,$$

$$y(0) = 0, \quad 2y(1) - \frac{dy}{dx}(1) = 0.$$

(b) (10 points) Plot these three eigenfunctions and verify that they have the correct number of zeros.

(c) (10 points) Verify the pairwise orthogonality of the three eigenfunctions.

(2) (60 points)

Consider the boundary value problem given below, which describes the transient heat flow in a slab of width  $L$  produced by a heat source  $\gamma(x,t)$ . The left end of the slab is insulated and the right end of the slab is maintained at zero temperature. The initial temperature is zero.

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} + \gamma(x,t), \quad 0 < x < L, \quad t > 0,$$

$$\text{with } \frac{\partial T}{\partial x}(0,t) = 0, \quad T(L,t) = 0, \quad \text{and } T(x,0) = 0.$$

You will solve this problem by expanding both  $T$  and  $\gamma$  in series of appropriate eigenfunctions.

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(2) (continued)

(a) (20 points) Find the appropriate eigenfunctions  $F_n(x)$  for this problem – i.e. the eigenfunctions of  $\partial^2/\partial x^2$  which satisfy the homogeneous boundary conditions at  $x = 0$  and  $L$ .

(b) (20 points) Look for solutions in the form  $T(x,t) = \sum_{n=1}^{\infty} C_n(t)F_n(x)$ . Use a similar expansion

for  $\gamma$ :  $\gamma(x,t) = \sum_{n=1}^{\infty} \gamma_n(t)F_n(x)$ . Substitute these expansions into the equation and obtain an ordinary differential equation to solve for each  $C_n(t)$ . Show that the proper initial conditions are  $C_n(0) = 0$ . In what circumstances would the initial conditions on  $C_n$  be non-zero?

(c) (20 points) Complete the solution when  $\gamma(x,t) = \gamma_0 e^{-\alpha t} \cos[(5\pi x)/(2L)]$ . Here  $\alpha$  and  $\gamma_0$  are positive constants. (You may assume that  $\alpha \neq \lambda_3 D$ .)

### CHALLENGE PROBLEM

1) (100 points) This problem is a continuation of the problem solved in section 5.1 of the class notes, and described in the Mathematica notebook entitled “Newton’s Law of Cooling.” In this continuation, you are asked to produce calculations and a graph for the heat flux leaving the slab at the right face. You are to use exactly the same formulation and parameter values as those in the Mathematica notebook. Probably the easiest way to do this problem is to add a section 10 to the given notebook, and to carry out the tasks given below in that section 10. If you do this, you need print out only your new section 10 to hand in.

(a) (30 points) Plot the heat flux leaving the right face (in  $W/m^2$ ) versus time in hours from 0 to 200 hours.

(b) (30 points) Find the total heat outflow in the first 200 hours from the right face (per unit slab area) by integrating the heat flux with respect to time. Do this both analytically (using Integrate) and numerically (using NIntegrate).

(c) (20 points) At any given time, the total energy in the slab, per unit face area, is

$$\int_0^L \rho C T(x,t) dx,$$

where  $\rho$  is the density and  $C$  is the specific heat. (Recall that  $D = k/(\rho C)$ , where  $D$  is the diffusivity and  $k$  is the thermal conductivity.) Calculate the initial thermal energy and the final steady state thermal energy per unit slab area.

(d) (20 points) Show that your calculations of parts (b) and (c) are consistent with energy conservation.