

Physics 5013. Homework 7  
Due Monday, November 20, 2006

November 7, 2006

**Problems in Bender and Orzag:**

Chapter 6, pp. 313-4: 6.71, 6.76

1. Show that

$$(2^s - 1)\zeta(s) = \frac{2^{s-1}s}{s-1} + 2 \int_0^\infty \left(\frac{1}{4} + y^2\right)^{-s/2} \sin(s \arctan 2y) \frac{dy}{e^{2\pi y} - 1}.$$

2. One representation for the gamma function is

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_C e^{t-t^x} dt$$

where the contour of integration is as shown in Figure 1. Use the method of steepest descents to derive Stirling's approximation,

$$\Gamma(x) \sim x^x e^{-x} \sqrt{\frac{2\pi}{x}}, \quad x \rightarrow +\infty.$$

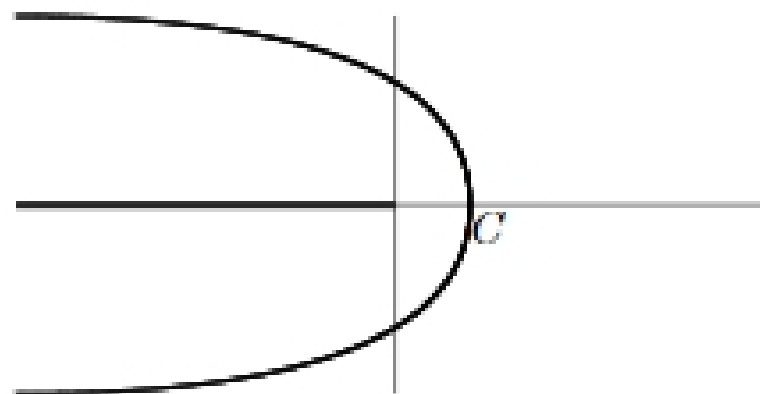


Figure 1: Contour  $C$  used for defining the Hankel representation of the gamma function.

3. The Airy function has the asymptotic expansion

$$\text{Ai}(x) \sim \frac{1}{2} \pi^{-1/2} x^{-1/4} e^{-2x^{3/2}/3} [1 + O(x^{-3/2}) + O(x^{-3}) + \dots].$$

Calculate the  $O(x^{-3/2})$  and the  $O(x^{-3})$  terms.

4. Using the integral representation for the Hankel function of the first kind

$$H_{\nu}^{(1)}(z) = \frac{e^{-i\nu\pi/2}}{\pi} \int_{-\pi/2+i\infty}^{\pi/2-i\infty} d\phi e^{i(z \cos \phi + \nu\phi)},$$

derive Debye's asymptotic expansion of  $\tan \alpha > 0$  and  $\nu$  large and positive:

$$H_{\nu}^{(1)}(\nu \sec \alpha) \sim \sqrt{\frac{2}{\pi \nu \tan \alpha}} e^{-i\pi/4} e^{i\nu(\tan \alpha - \alpha)} \left( 1 + \frac{u_1(\cot \alpha)}{i\nu} + O(1/\nu^2) \right),$$

where

$$u_1(t) = \frac{3t + 5t^3}{24}.$$