Lecture 7

Circular Motion: vector description

Position $\vec{r}(t) = r\,\hat{r}(t)$

Component of angular velocity $\omega_z = d\theta/dt$

Velocity $\vec{v} = v_\theta \hat{\theta}(t) = r\,(d\theta/dt)\,\hat{\theta}$

Component of Angular acceleration $\alpha_z = d\omega_z/dt = d^2\theta/dt^2$

Acceleration $a_r = -r\,(d\theta/dt)^2 = -v^2/r \qquad a_\theta = r\,d^2\theta/dt^2$

Polar coordinate system

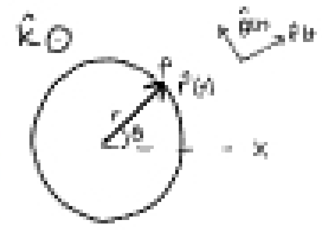
Coordinates (r, θ)

Unit vectors $(\hat{r}, \hat{\theta})$

Relation to Cartesian Coordinates

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}(y/x)$



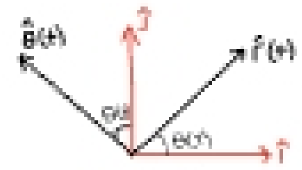
Transformations between unit vectors in polar coordinates and cartesian unit vectors

$\hat{r}(t) = \cos\theta(t)\,\hat{i} + \sin\theta(t)\,\hat{j}$

$\hat{\theta}(t) = -\sin\theta(t)\,\hat{i} + \cos\theta(t)\,\hat{j}$

$\hat{i} = \cos\theta(t)\,\hat{r}(t) - \sin\theta(t)\,\hat{\theta}(t)$

$\hat{j} = \sin\theta(t)\,\hat{r}(t) + \cos\theta(t)\,\hat{\theta}(t)$



Circular Motion: Position

Position Vector: $\vec{r}(t) = r\,\hat{r}(t) = r\,(\cos(\theta(t))\,\hat{i} + \sin(\theta(t))\,\hat{j})$

Velocity Vector: $\vec{v} = \frac{d\vec{r}}{dt} = -r\sin\theta \frac{d\theta}{dt}\,\hat{i} + r\cos\theta \frac{d\theta}{dt}\,\hat{j} \rightarrow \vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\theta}{dt}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \rightarrow \vec{v} = r\frac{d\theta}{dt}\,\hat{\theta}$

Fixed axis rotation: Angular Velocity

Angle variable: θ

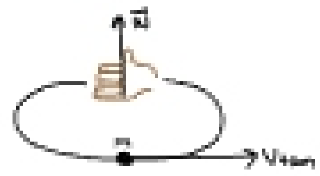
Angular velocity: $\vec{\omega} = d\theta/dt\ \hat{k}$

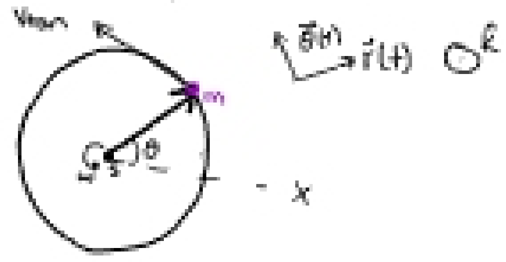
Component: $\omega_z = d\theta/dt$

Magnitude: $\omega = |\omega_z| = |d\theta/dt|$

Direction: $\omega_z = d\theta/dt > 0$, $+\hat{k}$ direction

$\omega_z = d\theta/dt < 0$, $-\hat{k}$ direction





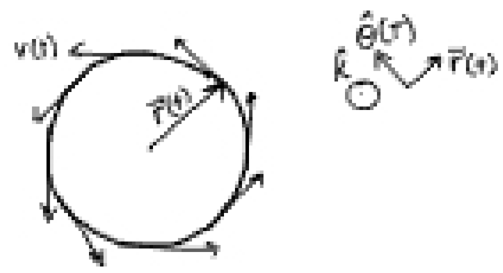
Speed and angular speed

The tangential component of the velocity of the object undergoing circular motion is proportional to the rate of change of the angle with time

$v_\theta = r\frac{d\theta}{dt} = r\omega$

Angular speed: $\omega = \frac{d\theta}{dt}$ (units: $rad \cdot s^{-1}$)

Velocity: $\vec{v}(t) = v_\theta \hat{\theta} = r\omega\,\hat{\theta}$



Circular Motion: Constant Speed, Period, and Frequency

In one period the object travels a distance equal to the circumference $s = 2\pi R = vT$

Period: the amount of time to complete one circular orbit of radius R

$T = \frac{2\pi R}{v} = \frac{2\pi R}{R\omega} = \frac{2\pi}{\omega}$

Frequency is the inverse of the period

$f = \frac{1}{T} = \frac{\omega}{2\pi}$ (units: s^{-1} or Hz)

Acceleration and circular motion

When an object moves in a circular orbit, the direction of the velocity changes and the speed may change as well

$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$

For circular motion, the acceleration will always have a non-positive radial component (a_r) due to the change in direction of velocity (it may be zero at the instant the velocity is zero)

Fixed axis rotation: Angular Acceleration

Angular acceleration: $\vec{\alpha} = \alpha_z \hat{k} = \frac{d^2\theta}{dt^2}\hat{k}$

Component: $\alpha_z = \frac{d^2\theta}{dt^2}$

Magnitude: $\alpha = |\alpha_z| = \frac{d^2\theta}{dt^2}$

Direction: $\alpha_z = d^2\theta/dt^2 > 0$ $+\hat{k}$ direction

$\alpha_z = d^2\theta/dt^2 < 0$ $-\hat{k}$ direction

Alternative forms of magnitude of centripetal acceleration

Parameters: speed *v*, angular speed *ω*, frequency *f*, and period T

$$|a_r| = \frac{v^2}{r} = r\omega^2 = r(2\pi f)^2 = \frac{4\pi^2 r}{T^2}$$

Circular Motion: Tangential Acceleration

When the component of the angular velocity is a function of time, $\omega_z(t) = \frac{d\theta}{dt}(t)$

The component of the velocity has a non-zero derivative $\frac{dv_\theta(t)}{dt} = r\frac{d^2\theta}{dt^2}(t)$

The the *tangential acceleration* is the time rate of change of the magnitude of the velocity $\vec{a}_\theta(t) = a_\theta(t)\hat{\theta}(t) = r\frac{d^2\theta}{dt^2}(t)\hat{\theta}(t) = r\alpha_z(t)\hat{\theta}(t)$