6.003: Signals and Systems

Fourier Transform

November 10, 2009

Mid-term Examination #3

Wednesday, November 18, 7:30-9:30pm, Walker Memorial
(this exam is after drop date).

No recitations on the day of the exam.

Coverage: cumulative with more emphasis on recent material
lectures 1–18
homeworks 1–10

Homework 10 will not collected or graded. Solutions will be posted.

Closed book: 3 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.

Conflict? Contact freeman@mit.edu by Friday, November 13, 2009.

Last Week: Fourier Series

Representing periodic signals as sums of sinusoids.

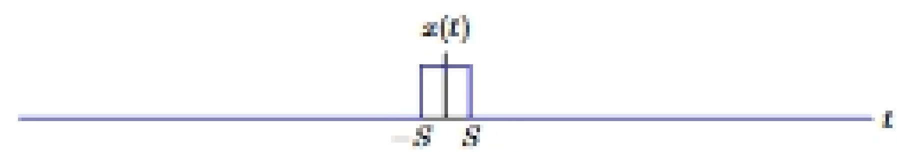
→ new representations for systems as filters.

This week: generalize for aperiodic signals.

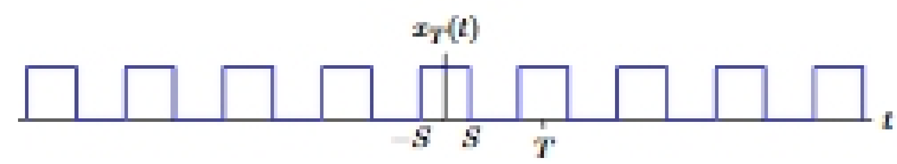
Fourier Transform

An aperiodic signal can be thought of as periodic with infinite period.

Let $x(t)$ represent an aperiodic signal.



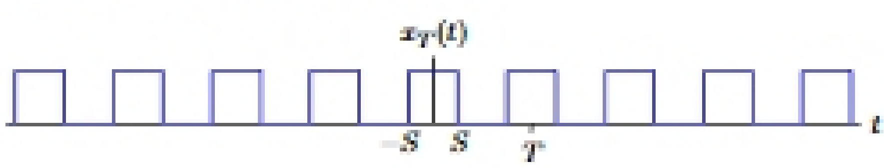
"Periodic extension": $x_T(t) = \sum_{k=-\infty}^{\infty} x(t+kT)$



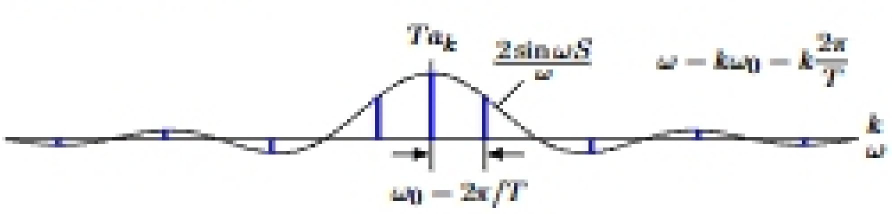
Then $x(t) = \lim_{T\to\infty} x_T(t)$.

Fourier Transform

Represent $x_T(t)$ by its Fourier series.

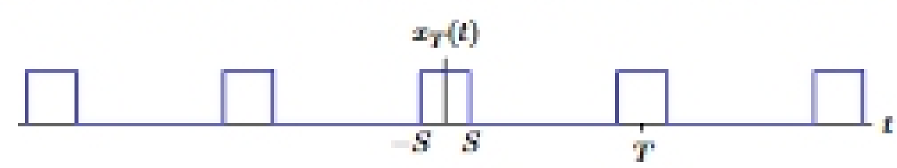


$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

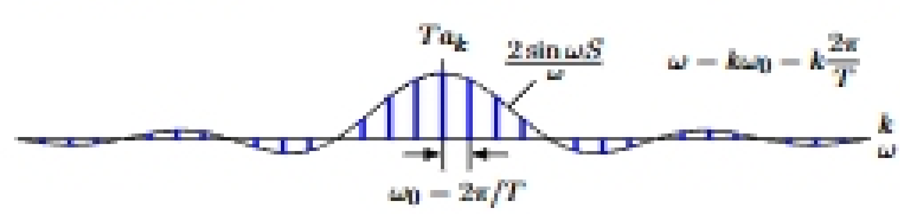


Fourier Transform

Doubling period doubles # of harmonics in given frequency interval.

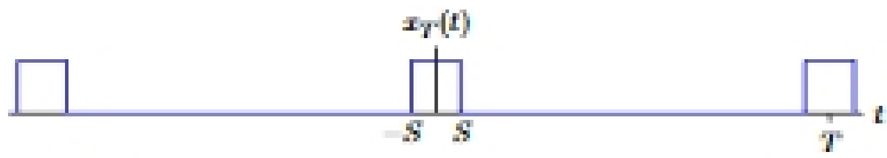


$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

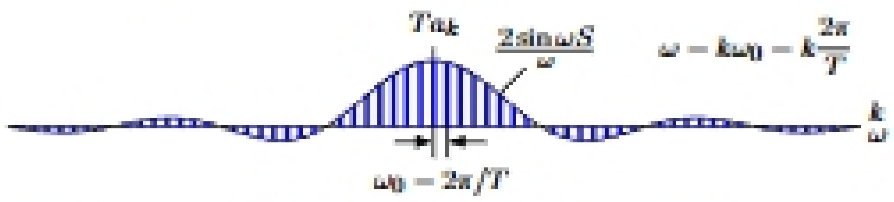


Fourier Transform

As $T \to \infty$, discrete harmonic amplitudes $\to$ a continuum $E(\omega)$.



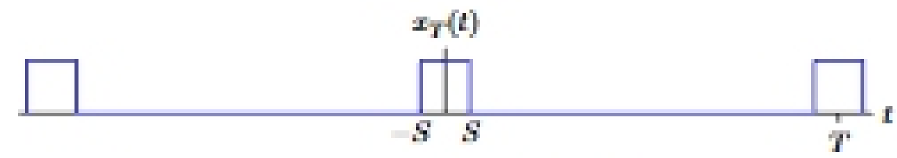
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

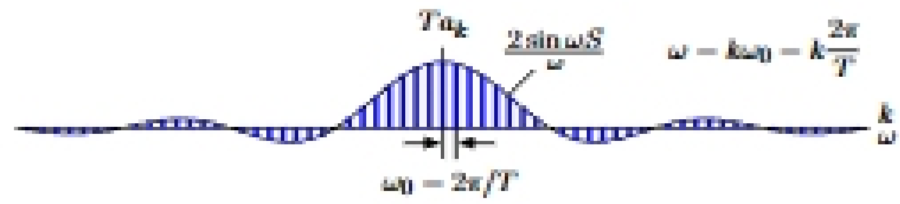


$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

Fourier Transform

As $T \to \infty$, synthesis sum $\to$ integral.





$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

$$x(t) = \sum_{k=-\infty}^{\infty}\underbrace{\frac{1}{T}E(\omega)}_{a_k}e^{j\frac{2\pi}{T}kt} = \sum_{k=-\infty}^{\infty}\frac{\omega_0}{2\pi}E(\omega)e^{j\omega t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty}E(\omega)e^{j\omega t}d\omega$$

Fourier Transform

Replacing $E(\omega)$ by $X(j\omega)$ yields the Fourier transform relations.

$$E(\omega) = X(s)\big|_{s=j\omega} \equiv X(j\omega)$$

Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{("analysis" equation)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{("synthesis" equation)}$$

Relation between Fourier and Laplace Transforms

If the Laplace transform of a signal exists and if the ROC includes the $j\omega$ axis, then the Fourier transform is equal to the Laplace transform evaluated on the $j\omega$ axis.

Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Fourier transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = H(s)\big|_{s=j\omega}$$

Relation between Fourier and Laplace Transforms

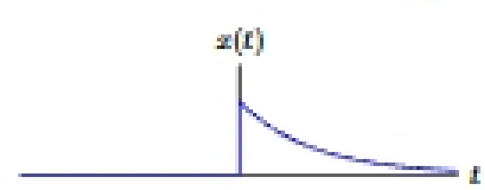
Fourier transform "inherits" properties of Laplace transform.

Property	$x(t)$	$X(s)$	$X(j\omega)$
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	$aX_1(j\omega)+bX_2(j\omega)$
Time shift	$x(t-t_0)$	$e^{-st_0}X(s)$	$e^{-j\omega t_0}X(j\omega)$
Time scale	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Differentiation	$\frac{dx(t)}{dt}$	$sX(s)$	$j\omega X(j\omega)$
Multiply by t	$tx(t)$	$-\frac{d}{ds}X(s)$	$-\frac{1}{j}\frac{d}{d\omega}X(j\omega)$
Convolution	$x_1(t)*x_2(t)$	$X_1(s)\times X_2(s)$	$X_1(j\omega)\times X_2(j\omega)$

Relation between Fourier and Laplace Transforms

There are also important differences.

Compare Fourier and Laplace transforms of $x(t) = e^{-t}u(t)$.



Laplace transform

$$X(s) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt = \int_0^{\infty} e^{-(s+1)t}dt = \frac{1}{1+s} \;;\; \text{Re}(s) > -1$$

a complex-valued function of **complex** domain.

Fourier transform

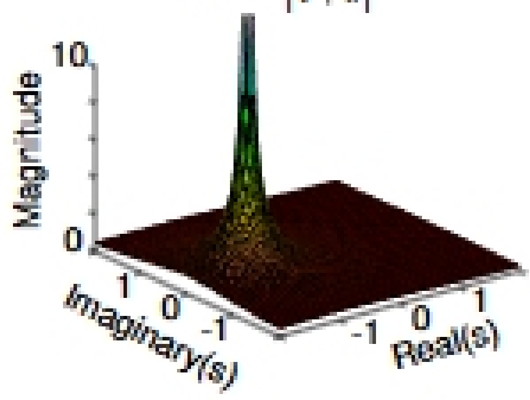
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-(j\omega+1)t}dt = \frac{1}{1+j\omega}$$

a complex-valued function of **real** domain.

Laplace Transform

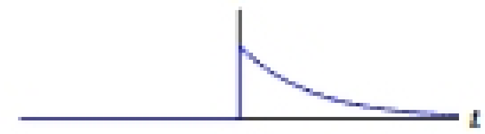
The Laplace transform maps a function of time t to a complex-valued function of complex-valued domain s.

$$|X(s)| = \left|\frac{1}{1+s}\right|$$



Fourier Transform

The Fourier transform maps a function of time t to a complex-valued function of real-valued domain ω.

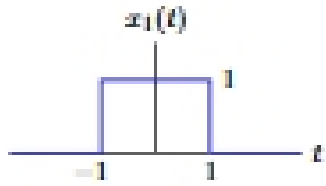


$$|X(j\omega)| = \left|\frac{1}{1+j\omega}\right|$$

Frequency plots promote intuition in ways not possible with s.

Check Yourself

Find the Fourier transform of the following square pulse.

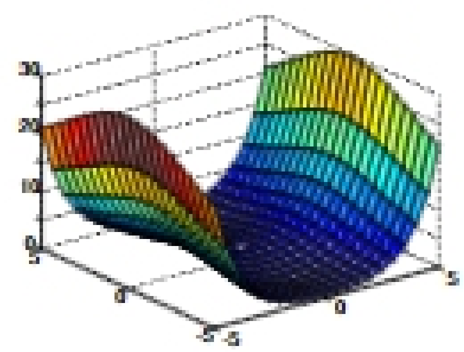


1. $X_1(j\omega) = \frac{1}{\omega}\left(e^{\omega} - e^{-\omega}\right)$
2. $X_1(j\omega) = \frac{1}{\omega}\sin\omega$
3. $X_1(j\omega) = \frac{2}{\omega}\left(e^{\omega} - e^{-\omega}\right)$
4. $X_1(j\omega) = \frac{2}{\omega}\sin\omega$
5. none of the above

Laplace Transform

Laplace transform: complex-valued function of complex domain.

$$|X_1(s)| = \left|\frac{1}{s}\left(e^{s} - e^{-s}\right)\right|$$

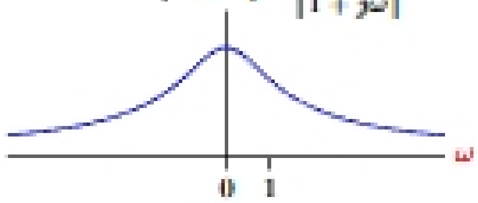


Fourier Transform

The Fourier transform is a function of real domain: frequency ω.

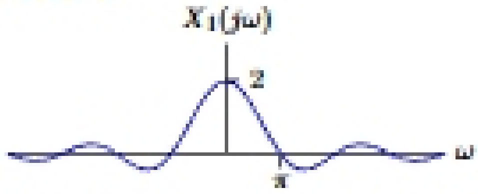
Time representation:

$$|X(j\omega)| = \left|\frac{1}{1+j\omega}\right|$$



Frequency representation:

$X_1(j\omega)$



Fourier Transform

One of the most useful features of the Fourier transform (and Fourier series) is the simple "inverse" Fourier transform.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{(Fourier transform)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{("inverse" Fourier transform)}$$