MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

TUTORIAL 2

Tuesday, February 17, 2004

Announcements

- Problem set 2 is due this Friday.
- There are no tutorials this week, but we will provide this handout, and TAs will hold extra office hours during the week.

Today's Agenda

- DT Convolution
  - Calculating the DT convolution
- CT Convolution
  - Calculating the CT convolution
- LTI System Properties
  - Impulse response
  - Commutativity, distributivity, associativity, and time shift
  - Effect of stability on commutativity
  - Causality and stability
- Signal Properties vs. System Properties
- Singularity Functions

1 DT Convolution

In class, we saw how to write any DT signal as the superposition sum of scaled, shifted impulses using the sifting property of the DT unit impulse:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

We define the *unit impulse response*, $h[n]$, of a DT system H as the output when the input is the unit impulse $\delta[n]$. If the system H is LTI, then the output $y[n]$ for any arbitrary input $x[n]$ is

$$\begin{aligned}
y[n] &= H\{x[n]\} \\
&= H\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\right\} \\
&= \sum_{k=-\infty}^{+\infty} x[k]\, H\{\delta[n-k]\} \quad \text{(by linearity)} \\
&= \sum_{k=-\infty}^{+\infty} x[k]\, \left(H\{\delta[n]\}_{\text{shifted by k}}\right) \quad \text{(by time-invariance)} \\
&= \sum_{k=-\infty}^{+\infty} x[k]\, \left(h[n]_{\text{shifted by k}}\right) \quad \text{(apply system } H\text{)} \\
y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad \text{(simplify)}
\end{aligned}$$

So, the output $y[n]$ is the *convolution* of the input $x[n]$ and the unit impulse response $h[n]$ of the LTI system. We write this as

$$y[n] = x[n] * h[n].$$

One useful sanity check when performing DT convolutions is that a signal of length m convolved with a signal of length n produces a signal with no more than $m+n-1$ non-zero terms.

1.1 Calculating the DT convolution

Calculating the convolution of two DT signals can be a rather tedious process. Consider the following example.

Given $x[n] = \delta[n+1] - 2\delta[n] + 3\delta[n-1]$ and $h[n] = \delta[n] + 4\delta[n-1] - 2\delta[n-2] + \delta[n-3]$, calculate $y[n] = x[n] * h[n]$. We can calculate the convolution by applying the convolution sum directly

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Since $x[n]$ is nonzero for $n \in \{-1, 0, 1\}$,

$$\begin{aligned}
y[n] &= \sum_{k=-1}^{1} x[k]h[n-k] \\
y[-1] &= x[-1]h[0] = 1 \\
y[0] &= x[-1]h[1] + x[0]h[0] = 4 + -2 = 2 \\
y[1] &= x[-1]h[2] + x[0]h[1] + x[1]h[0] = -2 + -8 + 3 = -7 \\
y[2] &= x[-1]h[3] + x[0]h[2] + x[1]h[1] = 1 + 4 + 12 = 17 \\
y[3] &= x[0]h[3] + x[1]h[2] = -2 + -6 = -8 \\
y[4] &= x[1]h[3] = 3
\end{aligned}$$

Thus, $y[n] = \delta[n+1] + 2\delta[n] - 7\delta[n-1] + 17\delta[n-2] - 8\delta[n-3] + 3\delta[n-4]$.

It is sometimes quicker, and less error prone, to use the following trick when calculating the convolution of two *finite length DT signals*. First we form the following table:

$x[n] \diagdown h[n]$	$\underline{1}$	4	-2	1
1	1	4	-2	1
$\underline{-2}$	$\underline{\underline{-2}}$	-8	4	-2
3	3	12	-6	3

The table consists of the values of $x[n]$ in the first column and the values of $h[n]$ in the top row. The elements in the interior of the table are formed by multiplying the corresponding elements in the first column and first row. Note that the *underlined* entries correspond to the element at zero (i.e. $x[0]$ and $h[0]$). The *double-underlined* element corresponds to the entry formed by multiplying the two underlined elements (i.e. $x[0]h[0]$). Now we can simply form $y[n]$ by summing diagonally. Using sequence notation:

$$\begin{aligned}
y[n] &= \Big\{1,\ \underline{4 + (-2)},\ (-2) + (-8) + 3,\ 1 + 4 + 12,\ (-2) + (-6),\ 3\Big\} \\
&= \{1,\ \underline{2},\ -7,\ 17,\ -8,\ 3\}
\end{aligned}$$

Note again that we have underlined the element resulting from a sum containing the double-underlined element, which corresponds to $y[0]$. This result for $y[n]$ is equivalent to the one above.