MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

TUTORIAL 4

Monday, March 1 and Tuesday, March 2, 2004

Announcements

- Problem set 4 is due this Friday.
- Quiz 1 will be held on **Thursday, March 11**, 7:30–9:30 p.m. in Walker Memorial. The quiz will cover material in Chapters 1–3 of O&W, Lectures and Recitations through Feburary 27, Problem Sets #1–3, and that part of Problem Set #4 involving problems from Chapter 3.
- The TAs will jointly hold office hours from 2–8 p.m. on Wednesday, March 10 and again from 10 a.m.–3 p.m. on Thursday, March 11. A schedule will be posted on the 6.003 website.
- A quiz review package will be available on the 6.003 website this Thursday. TAs will hold two identical optional quiz review sessions on Monday, March 8 and Tuesday, March 9, 7:30–9:30 p.m. in 34-101.

Today's Agenda

- Frequency Response of LTI Systems
  - Differential and difference equations
  - Filtering
  - Real systems
  - Frequency response of cascaded systems
- CT Fourier Transform
  - Synthesis and analysis equations
  - Variations of the synthesis and analysis equations
  - Rectangular pulse and sinc pair
  - The multiplication and convolution properties

1 Frequency Response of LTI Systems

In our Fourier series representation of periodic signals, we set the CT variable $s = j\omega$, so that e^{st} becomes $e^{j\omega t}$. Likewise, in DT, we set $z = e^{j\omega}$, so that z^n becomes $e^{j\omega n}$. Then, the eigenvalue of the LTI system corresponding to the eigenfunction $e^{j\omega t}$ (CT) and $e^{j\omega n}$ (DT) is the *frequency response* of a system H. It is defined through the impulse response $h(t)$ (CT) and $h[n]$ (DT) as:

The Frequency Response of LTI Systems in Terms of the Impulse Response:

$$
\begin{aligned}
H(j\omega) &= \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}\,\mathrm{d}t \qquad \text{(CT)} \\
H(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} \qquad \text{(DT)}
\end{aligned}
$$

From the eigenfunction property, when these exponentials are the inputs of an LTI system, the outputs are the same exponentials scaled by the frequency response of the system. Now that we know how to write periodic input signals as the linear combination of complex exponentials by determining the Fourier series coefficients, we can scale the coefficients appropriately according to the frequency response of the system to get the Fourier series coefficients of the output. So, if the inputs are periodic signals with Fourier series coefficients a_k:

$$
\begin{aligned}
x(t) &= \sum_k a_k e^{jk\omega_0 t} \qquad \text{(CT)} \\
x[n] &= \sum_k a_k e^{jk\omega_0 n} \qquad \text{(DT)}
\end{aligned}
$$

then the outputs are periodic signals with Fourier series coefficients $b_k = H(jk\omega_0)a_k$ for CT and $b_k = H(e^{jk\omega_0})a_k$ for DT:

$$
\begin{aligned}
y(t) &= \sum_k a_k H(jk\omega_0) e^{jk\omega_0 t} \qquad \text{(CT)} \\
y[n] &= \sum_k a_k H(e^{jk\omega_0}) e^{jk\omega_0 n} \qquad \text{(DT)}
\end{aligned}
$$

There is a caveat when speaking about the frequency response of LTI systems. All stable systems have well-defined frequency responses for all frequencies. However, unstable systems generally do not have a frequency response.

1.1 Differential and difference equations

A large number of LTI systems that we study in real life are described by linear constant-coefficient ordinary differential (CT) and difference (DT) equations (LCCODEs), so it would be helpful to develop techniques to analyze such systems. As we found in problem set 2, finding the impulse response of such systems (*time-domain* analysis) is a rather tedious procedure. However, it turns out that a *frequency-domain* analysis is much more straightforward:

Finding the Frequency Response of Differential and Difference Equations:

Suppose we are given a *stable* CT or DT system described by a differential or difference equation. To find the frequency response, we do the following:

1. Let $x(t) = \mathrm{e}^{j\omega t}$ for CT or $x[n] = \mathrm{e}^{j\omega n}$ for DT.
2. Let $y(t) = H(j\omega)\mathrm{e}^{j\omega t}$ for CT or $y[n] = H(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}$ for DT.
3. Plug $x(t)$ and $y(t)$ for CT or $x[n]$ and $y[n]$ for DT into the differential or difference equation.
4. Solve for $H(j\omega)$ for CT or $H(\mathrm{e}^{j\omega})$ for DT.

If we apply this method, we get the following result.

The Frequency Response of Differential and Difference Equations:

Suppose we are given a *stable* CT system described by the following differential equation:

$$\begin{aligned} &a_N \frac{\mathrm{d}^N}{\mathrm{d}t^N} y(t) + a_{N-1} \frac{\mathrm{d}^{N-1}}{\mathrm{d}t^{N-1}} y(t) + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}t} y(t) + a_0 y(t) \\ &= b_M \frac{\mathrm{d}^M}{\mathrm{d}t^M} x(t) + b_{M-1} \frac{\mathrm{d}^{M-1}}{\mathrm{d}t^{M-1}} x(t) + \cdots + b_1 \frac{\mathrm{d}}{\mathrm{d}t} x(t) + b_0 x(t). \end{aligned}$$

Its frequency response is:

$$H(j\omega) = \frac{b_M (j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_1(j\omega) + a_0}{a_N (j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0}$$

Similarly, a *stable* DT system described by the following difference equation:

$$\begin{aligned} &a_N y[n-N] + a_{N-1} y[n-(N-1)] + \cdots + a_1 y[n-1] + a_0 y[n] \\ &= b_M x[n-M] + b_{M-1} x[n-(M-1)] + \cdots + b_1 x[n-1] + b_0 x[n], \end{aligned}$$

has frequency response:

$$H(\mathrm{e}^{j\omega}) = \frac{b_M \mathrm{e}^{-jM\omega} + b_{M-1}\mathrm{e}^{-j(M-1)\omega} + \cdots + b_1 \mathrm{e}^{-j\omega} + b_0}{a_N \mathrm{e}^{-jN\omega} + a_{N-1}\mathrm{e}^{-j(N-1)\omega} + \cdots + a_1 \mathrm{e}^{-j\omega} + a_0}$$