# 6.003: Signals and Systems

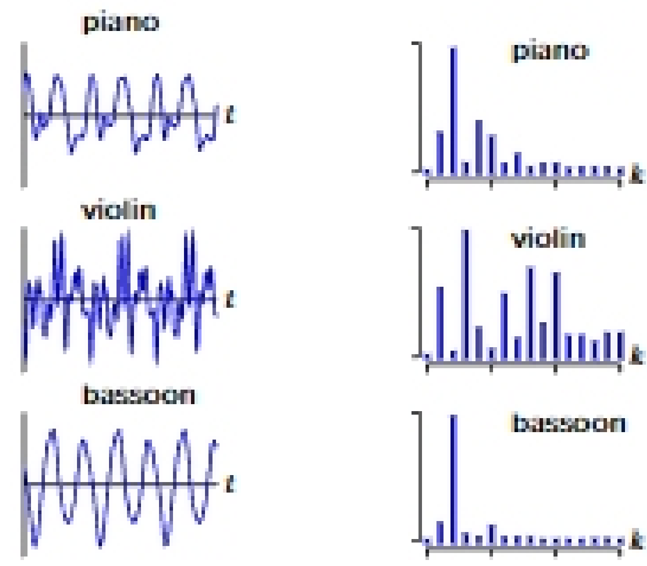
## Fourier Series

*November 5, 2009*

## Last Time: Describing Signals by Frequency Content

Harmonic content is natural way to describe some kinds of signals.

Ex: musical instruments (http://theremin.music.uiowa.edu/MIS)




## Last Time: Fourier Series

Determining harmonic components of a periodic signal.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt \qquad \text{("analysis" equation)}$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \qquad \text{("synthesis" equation)}$$

## Separating harmonic components

Underlying properties.

1. Multiplying two harmonics produces a new harmonic with the same fundamental frequency:

$$e^{jk\omega_0 t} \times e^{jl\omega_0 t} = e^{j(k+l)\omega_0 t}.$$

2. The integral of a harmonic over any time interval with length equal to a period $T$ is zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_0 t}dt = \int_T e^{jk\omega_0 t}dt = \begin{cases} 0, & k \neq 0 \\ T, & k = 0 \end{cases}$$
$$= T\delta[k]$$

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Closure: the set of harmonics is closed under multiplication.

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Orthogonality: harmonics are orthogonal ($\perp$) to each other.

## Fourier Series as Orthogonal Decompositions

Analogy with vectors in 3-space.

Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ represent direction vectors in 3-space.

Vector $\vec{r}$ can be expressed as sum of components $x\hat{x} + y\hat{y} + z\hat{z}$ where

$$x = \vec{r}\cdot\hat{x}$$
$$y = \vec{r}\cdot\hat{y}$$
$$z = \vec{r}\cdot\hat{z}$$

Similarly for Fourier series (where basis functions are $\phi_k(t) = e^{j\frac{2\pi}{T}kt}$), a signal can be expressed as a sum of orthogonal components:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\phi_k(t)$$

where the coefficient of each component is a dot product

$$a_k = x(t)\cdot\phi_k(t) = \frac{1}{T}\int_T x(t)\phi_k^*(t)dt$$

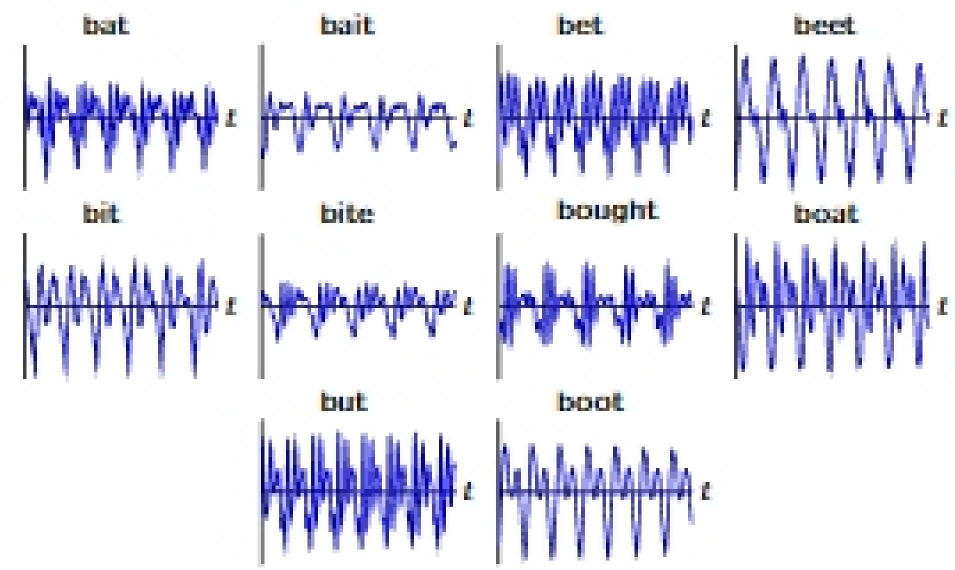
## Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$?

1. $\cos 2\pi t \perp \sin 2\pi t$ ?
2. $\cos 2\pi t \perp \cos 4\pi t$ ?
3. $\cos 2\pi t \perp \sin \pi t$ ?
4. $\cos 2\pi t \perp e^{j2\pi t}$ ?

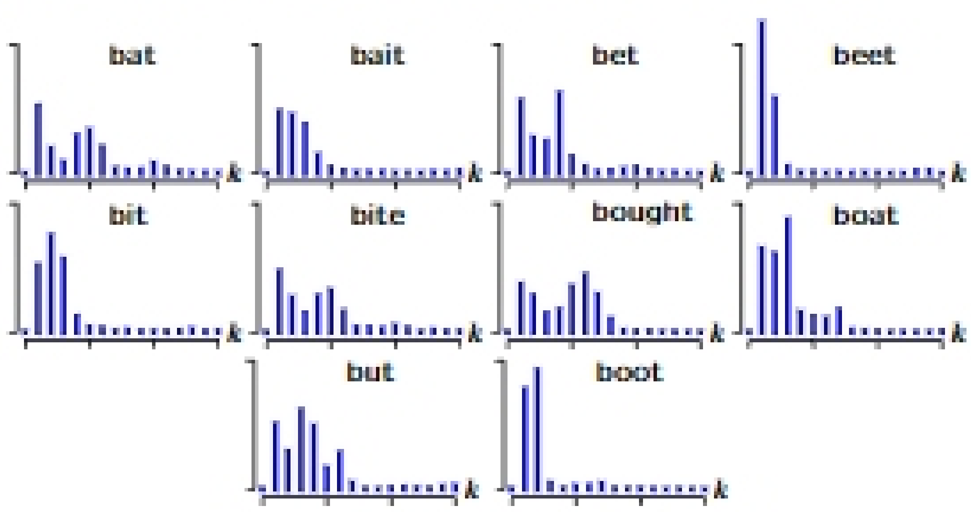
## Speech

Vowel sounds are quasi-periodic.




## Speech

Harmonic content is natural way to describe vowel sounds.

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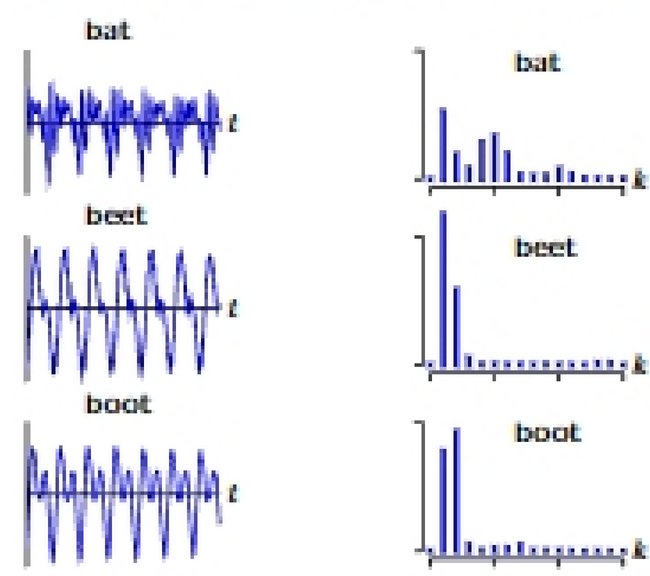
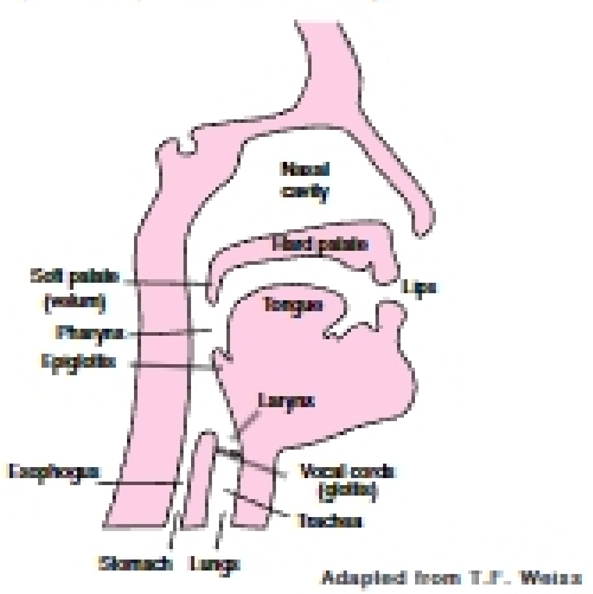



## Speech Production

Speech is generated by the passage of air from the lungs, through the vocal cords, mouth, and nasal cavity.

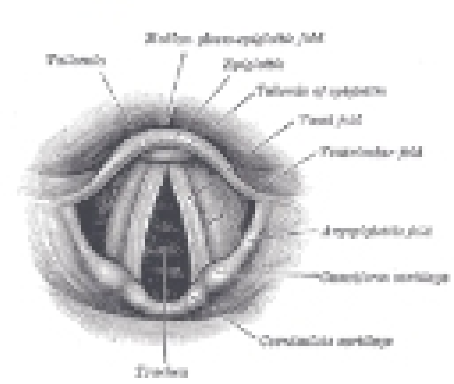



Adapted from T.F. Weiss

## Speech Production

Controlled by complicated muscles, the vocal cords are set into vibrational motion by the passage of air from the lungs.

Looking down the throat:

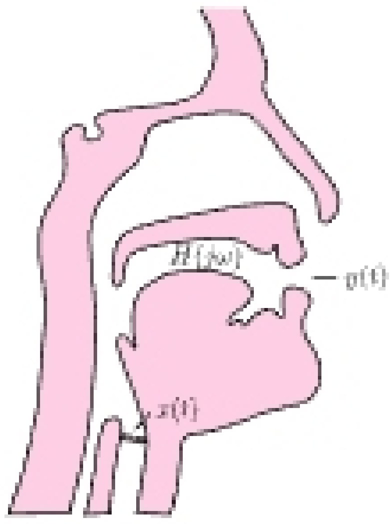


Gray's Anatomy


Adapted from T.F. Weiss

## Speech Production

Vibrations of the vocal cords are "filtered" by the mouth and nasal cavities to generate speech.



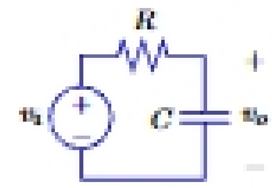

## Filtering

Notion of a filter.

LTI systems

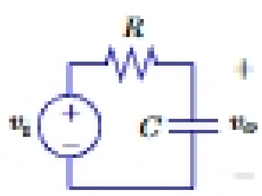
- cannot create new frequencies.
- can only scale magnitudes and shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit




## Lowpass Filter

Calculate the frequency response of an RC circuit.



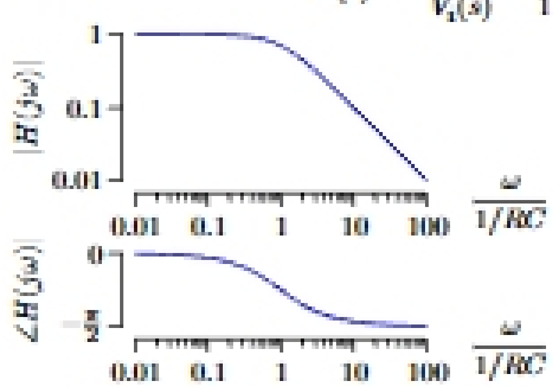

KVL: $v_i(t) = Ri(t) + v_o(t)$

C: $i(t) = C\dot{v}_o(t)$

Solving: $v_i(t) = RC\dot{v}_o(t) + v_o(t)$

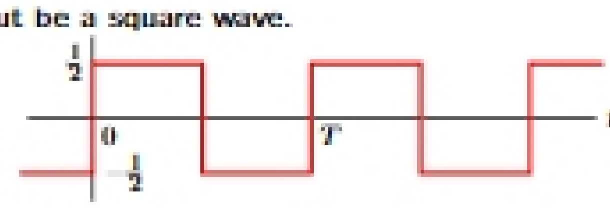
$$V_i(s) = (1 + sRC)V_o(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}$$

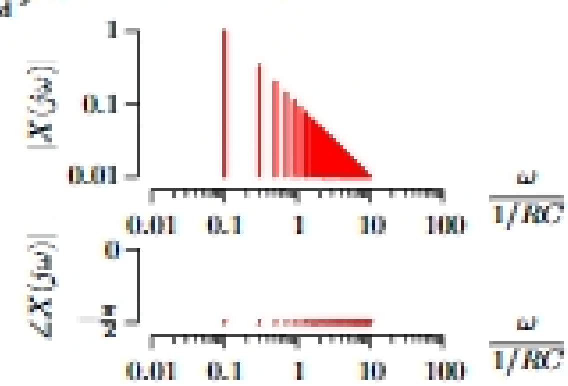



## Lowpass Filtering

Let the input be a square wave.

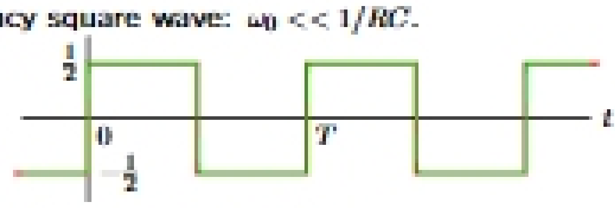



$$x(t) = \sum_{k\ \text{odd}} \frac{1}{j\pi k} e^{j\omega_0 kt}\ ; \quad \omega_0 = \frac{2\pi}{T}$$




## Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

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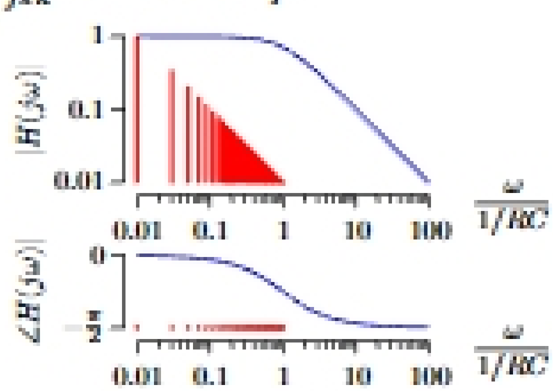
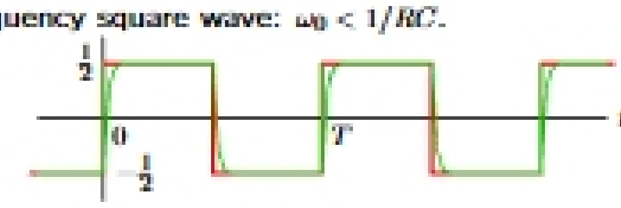



## Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

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