# Physics 5013. Homework 6
# Due Wednesday, November 16, 2011

November 4, 2011

1. Evaluate the following divergent sums using both the Euler and the generic methods:

   (a)
   $$1+0-1+0-1+0+1+0+1+0-1+\ldots,$$

   (b)
   $$1+0+0-1+1+0+0-1+1+0+0+\ldots,$$

   (c)
   $$1-1+0+0+1-1+0+0+\ldots.$$

2. Evaluate the following series using both the Euler and the Borel methods:
   $$\sum_{n=0}^{\infty}(-1)^n n^2.$$

3. Show that $0!-2!+4!-6!+\ldots$ is not Borel summable but that $0!+0-2!+0+4!+0-6!+0+\ldots$ is. Compute the Borel sum of the latter.

4. Consider the function
   $$\frac{1}{z}\log(1+z).$$
   Derive the $[3,3]$ Padé approximant stated in class
   $$P_3^3(z)=\frac{1+\frac{17}{14}z+\frac{1}{3}z^2+\frac{1}{140}z^3}{1+\frac{12}{7}z+\frac{6}{7}z^2+\frac{4}{35}z^3}.$$
   Similarly, work out $P_4^3(z)$, and verify the values given in class for $z=0.5$, 1, and 2.

5. The Stirling series for the Gamma function is for $n \to \infty$,

$$\Gamma(n) = (n-1)! \sim \left(\frac{2\pi}{n}\right)^{1/2} \left(\frac{n}{e}\right)^n \left(1 + \frac{A_1}{n} + \frac{A_2}{n^2} + \frac{A_3}{n^3} + \frac{A_4}{n^4} + \ldots\right),$$

where

$$\begin{aligned} A_1 &= \frac{1}{12}, \\ A_2 &= \frac{1}{288}, \\ A_3 &= -\frac{139}{51840}, \\ A_4 &= -\frac{571}{2488320}. \end{aligned}$$

Compute the $[1,1]$ and $[2,2]$ Padé approximants for $\Gamma(x)(e/x)^x\sqrt{x/2\pi}$. Compare numerically the values so obtained with the exact function for $x = 0.2$, $0.5$, and $1.0$, which are *small* values of $x$. Can a more accurate approximation be obtained by averaging $P_1^1$ and $P_2^2$?

6. Compute the first three terms of the continued-fraction coefficients of the series:

   (a)
   $$\sum_{n=0}^{\infty} (n!)^2(-z)^n,$$

   (b)
   $$\sum_{n=0}^{\infty} \frac{(-z)^n}{(2n)!},$$

   (c)
   $$\sum_{n=0}^{\infty} \frac{z^n}{n^2+1}.$$

7. Consider a continued-fraction representation of the exponential function in the form
$$e^x = \frac{c_0}{1 + \frac{c_1 x}{1 + \frac{c_2 x}{1 + \frac{c_3 x}{1 + \ldots}}}}.$$

Show that

$$c_0 = -c_1 = 1, \quad c_{2n} = \frac{1}{4n-2}, \quad c_{2n+1} = -\frac{1}{4n+2}, \quad n \geq 1.$$

How many terms must be included to compute $e$ to 8 significant figures?