Physics 5013. Homework 6
Due Friday, October 22, 2004

October 12, 2004

Problems in Whittaker and Watson:

Chapter 6, pp. 122–3: **2, 4, 5, 8, 9, 15**

Additional problems:

1. Evaluate
$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}(1+x^2)^2}$$
by using a contour which encircles the branch line given in Problem 5.3, and closed by a circle at infinity. Equivalently, consider a contour of two parts: one that just encloses the branch line from $z=-1$ to $z=+1$ and another being a circle about the origin of very large radius. Between these two contours, the function
$$f(z)=\sqrt{1-z^2}$$
is analytic.

2. Recall the generating function defining the Bernoulli numbers:
$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n.$$
Show that
$$B_n=\frac{n!}{2\pi i}\oint_{C_0}\frac{z}{e^z-1}\frac{dz}{z^{n+1}},$$
where C_0 is a circle about the origin with radius $|z|<2\pi$. From this integral find B_0, B_1 directly. By distorting C_0 into C, an infinite circle

about the origin (and hence crossing an infinite number of poles!), show that for n even, $n \geq 2$,

$$B_n = -\frac{(-1)^{n/2} 2\, n!}{(2\pi)^n} \zeta(n),$$

where

$$\zeta(n) = \sum_{p=1}^{\infty} p^{-n}.$$

3. Use the residue theorem to evaluate the following integrals:

   (a)
   $$\int_0^{2\pi} \frac{d\theta}{1 + a \sin^2 \theta}, \quad a > 0.$$

   (b)
   $$\int_0^{\infty} \frac{dx\, \sin x}{x(x^2 + a^2)}, \quad a > 0.$$

   (c)
   $$\int_0^{\infty} \frac{x^{2a-1}}{1 + x^2}, \quad 0 < a < 1.$$

   (d)
   $$P \int_0^{\infty} dx\, \frac{\sqrt{x}}{x^2 - a^2}, \quad a > 0.$$