Worksheet #6

I. Improper Integrals

1. Which of the following integrals is improper?

A. $\int_0^1 \frac{1}{x}\,dx$

B. $\int_2^3 \frac{1}{x}\,dx$

C. $\int_1^5 \frac{1}{x^2-6x}\,dx$

D. $\int_0^\infty \frac{1}{x^2+1}\,dx$

E. $\int_1^5 \frac{1}{x^2-4x+3}\,dx$

F. $\int_{-\infty}^3 \frac{2}{x-5}\,dx$

G. $\int_0^4 \frac{3}{x^2-5x-6}\,dx$

H. $\int_0^2 e^{-5x}\,dx$

2. Which of the following integrals is improper?

A. $\int_1^5 \sin\frac{1}{x}\,dx$

B. $\int_{-1}^1 |x|\,dx$

C. $\int_0^6 \frac{1}{x^3+x}\,dx$

D. $\int_0^\infty xe^{-x^2}\,dx$

E. $\int_0^1 \frac{6x-1}{x^3-x}\,dx$

F. $\int_{-\infty}^6 \frac{1}{x^2-5x+4}\,dx$

G. $\int_0^1 e^{-x^2}\,dx$

H. $\int_1^6 x\ln(5x-4)\,dx$

3. Given that:

$$\frac{2x+14}{(x-3)(x^2+1)} = \frac{2}{x-3} - \frac{2x+4}{x^2+1}$$

calculate the following or state the integral diverges.

a) $\int_0^1 \frac{2x+14}{(x-3)(x^2+1)}\,dx$

b) $\int_0^3 \frac{2x+14}{(x-3)(x^2+1)}\,dx$

c) $\int_0^5 \frac{2x+14}{(x-3)(x^2+1)}\,dx$

d) $\int_5^\infty \frac{2x+14}{(x-3)(x^2+1)}\,dx$

4. Given that:

$$\frac{5x^2+3x-1}{(x+2)(x^2+9)} = \frac{1}{x+2} + \frac{4x-5}{x^2+9}$$

Calculate the following or state that the integral diverges.

a) $\int_0^1 \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx$

b) $\int_{-2}^1 \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx$

c) $\int_{-5}^1 \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx$

d) $\int_{-\infty}^{-4} \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx$

State whether the following converge or diverge. If an integral converges, state the value to which it converges.

5. $\int_4^\infty \frac{4}{x^2-4}\,dx$

6. $\int_{10}^\infty \frac{3x}{x^2-9}\,dx.$

7. $\int_2^3 \frac{5x}{\sqrt{x^2-4}}\,dx$

8. $\int_{-1}^1 \frac{5x+1}{4\sqrt[3]{x}}\,dx.$

9. $\int_1^\infty \frac{4x^2+18}{x^3+9x}\,dx$

10. $\int_1^\infty \frac{60x^2-4x+100}{(x+1)(16x^2+25)}\,dx$

11. True or False:

a) $\int_{-1}^1 \frac{1}{x^2}\,dx = \int_{-1}^1 x^{-2}\,dx = \left[-x^{-1}\right|_{-1}^1 = \left[-\frac{1}{x}\right|_{-1}^1 = -2.$

b) $\int_{-1}^1 \frac{1}{(x+2)^2}\,dx = \int_{-1}^1 (x+2)^{-2}\,dx = \left[-(x+2)^{-1}\right|_{-1}^1 = -\frac{1}{x+2}\Big|_{-1}^1 = -\frac{1}{3}+1$

12. a) What does $\int_a^b f(x)\,dx$ represent?

b) Can we always use an antiderivative to evaluate it?

13. a) Show $\int_0^b \frac{1}{x^2+9}\,dx = \frac{1}{3}\arctan\frac{b}{3}$. Conclude $\int_0^\infty \frac{1}{x^2+9}\,dx = \frac{\pi}{6}$.

b) Using a computer program, such as Excel, make a list of the values of $\frac{1}{3}\arctan\frac{b}{3}$ for $b = 1, 2, 3, \ldots 1000$.

Do those values approach $\frac{\pi}{6}$ as b grows large?

II. Sequences

14. List the first 6 terms in the following sequences. $\{a_n\}_{n=1}$.

a) $a_n = 6n^2 - 13n$

b) $a_n = \sin \frac{n\pi}{2}$

c) $a_n = 4 - \frac{5}{n}$

d) $a_n = 3a_{n-1} + 1, \quad a_1 = 3$

e) $a_n = \frac{4a_{n-1}}{3a_{n-2}}$

f) $a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1, a_2 = 1$

15. Compute the limit of the following sequences, or state that they diverge. In each case below, use a computer program to calculate the first 1000 (or more) terms and notice what you find!

a) $a_n = \frac{3n^2 - 1}{\sqrt{n} + n + n^3}$

b) $a_n = \cos\left(\frac{1}{n^3}\right)$

c) $a_n = \frac{3\sin n^2 - n}{2n+1}$

d) $a_n = \frac{6n^2\cos n + 1}{5n - 4n^2}$

e) $a_n = \frac{\sqrt{9n^2+1}}{2n+3}$

f) $a_n = n^{\sin(\frac{1}{n})}$

16. Suppose a_n converges to 3, b_n converges to 6, and c_n diverges. State whether the following converge, diverge, or cannot be determined. If a sequence converges, state its value. If convergence cannot be determined, explain why!

a) $A_n = a_n + 3b_n$

b) $A_n = \frac{3a_n}{c_n}$

c) $A_n = \sqrt{c_n}$

d) $A_n = \sqrt{a_n + 4}$

e) $A_n = e^{b_n - 2}$

f) $A_n = a_n - 4c_n$

17. Suppose a_n converges, b_n converges, and c_n diverges, and d_n diverges. State whether the following are true or false. If a statement is false, provide a counterexample.