EE503 HW13 Solution
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Problem 1 [10 + 5 pt]
solution:
The CDF of X_n can be written as

$$\begin{aligned}
F_{X_n}(x) &= \int_{-\infty}^{x} f_{X_n}(t)dt \\
&= \begin{cases} 0 & x < 0 \\ \int_0^x 1 - \cos(2\pi n t)dt & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \\
&= \begin{cases} 0 & x < 0 \\ x - \frac{\sin(2\pi n x)}{2\pi n} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

which is the CDF of uniform R.V. between $[0, 1]$. Therefore, X_n converges to $U[0, 1]$ in distribution. If we take limits of densities then,

$$\lim_{n\to\infty} f_{X_n}(x) = 1 - \lim_{n\to\infty} \cos(2\pi n x)$$

which does not exist, therefore probability densities fail to converge in here.

Problem 2 [20 pt]
solution:
Since distribution function uniquely determine characteristic function (CHF), therefore the key idea would be to create sequence of CDF such that they converge to some function which is not a valid CDF. Consecutively, we will have sequence of CHF which converge to invalid CHF. The reason we are selecting CDF first and then computing CHF being, we would be sure that the CHF expression we have is valid. Otherwise if we denote any random function as CHF then we have to take inverse fourier transform to verify that CHF is valid from all perspective i.e. it gives a valid pdf.

Simplest example one can imagine is uniform R.V. Let us write a sequence of R.V. pdf as

$$f_{X_n}(x) = \begin{cases} \frac{1}{n} & 0 < x < n \\ 0 & o/w \end{cases}$$

The CDF would be

$$F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{n} & 0 \leq x < n \\ 1 & x \geq n \end{cases}$$

As $n \to \infty$ we can see that F_{X_n} would be 0 everywhere which is not possible. Let us now write CHF for this sequence of R.V.

$$\begin{aligned} \phi_{X_n}(t) &= \int_0^n e^{jtx}\frac{1}{n}dx \\ &= \frac{e^{jtn}-1}{jtn} \\ &= e^{\frac{jtn}{2}}\frac{sin(\frac{nt}{2})}{\frac{nt}{2}} \end{aligned}$$

Again, if we make $n \to \infty$ then

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi(t) = 0$$

which is not a valid CHF as expected. Now one can play with all well-known pdfs to create such cases, for example $X_n \sim exp(1/n)$ or $X_n \sim \mathcal{N}(0, 1/n)$ and so on.

Problem 3 [10 + 15 + 10 pt]
solution:

a) A simple application of Markov's Inequality

$$\begin{aligned}
\mathbb{P}(|X_n| > \epsilon) &\leq \frac{\mathbb{E}[|X_n|]}{\epsilon} \quad \forall n \\
\lim_{n\to\infty} \mathbb{P}(|X_n| > \epsilon) &\overset{(a)}{\leq} \lim_{n\to\infty} \frac{\mathbb{E}[|X_n|]}{\epsilon} \\
\lim_{n\to\infty} \mathbb{P}(|X_n| > \epsilon) &\leq 0 \\
\Rightarrow \lim_{n\to\infty} \mathbb{P}(|X_n| > \epsilon) &= 0
\end{aligned}$$

where we can write (a) because previous inequality is valid for all n. Therefore, X_n converges to 0 in probability.

b) We say X_n converges to X almost surely (a.s.) if

$$\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1$$

or in other words for any positive ϵ and N, X_n deviates from X at only finitely many places i.e.

$$\mathbb{P}(\lim_{n\to\infty} |X_n - X| \leq \epsilon \quad \forall n > N) = 1$$

or we can write $\mathbb{P}(|X_n - X| > \epsilon \ i.o.) = 0$

i) For this part, define sequence of R.V. as

$$X_n(\omega) = \begin{cases} 1 & 0 \leq \omega \leq \frac{1}{n} \\ 0 & \frac{1}{n} < \omega \leq 1 \end{cases}$$

and define the probability measure as simple counting measure i.e. length of the interval of ω set. Therefore $\mathbb{P}(X_n = 1) = \frac{1}{n}$. Now, to check whether X_n converges to 0 almost surely we have to see whether

$$\mathbb{P}(|X_n| > \epsilon \ i.o.) = 0 \tag{1}$$