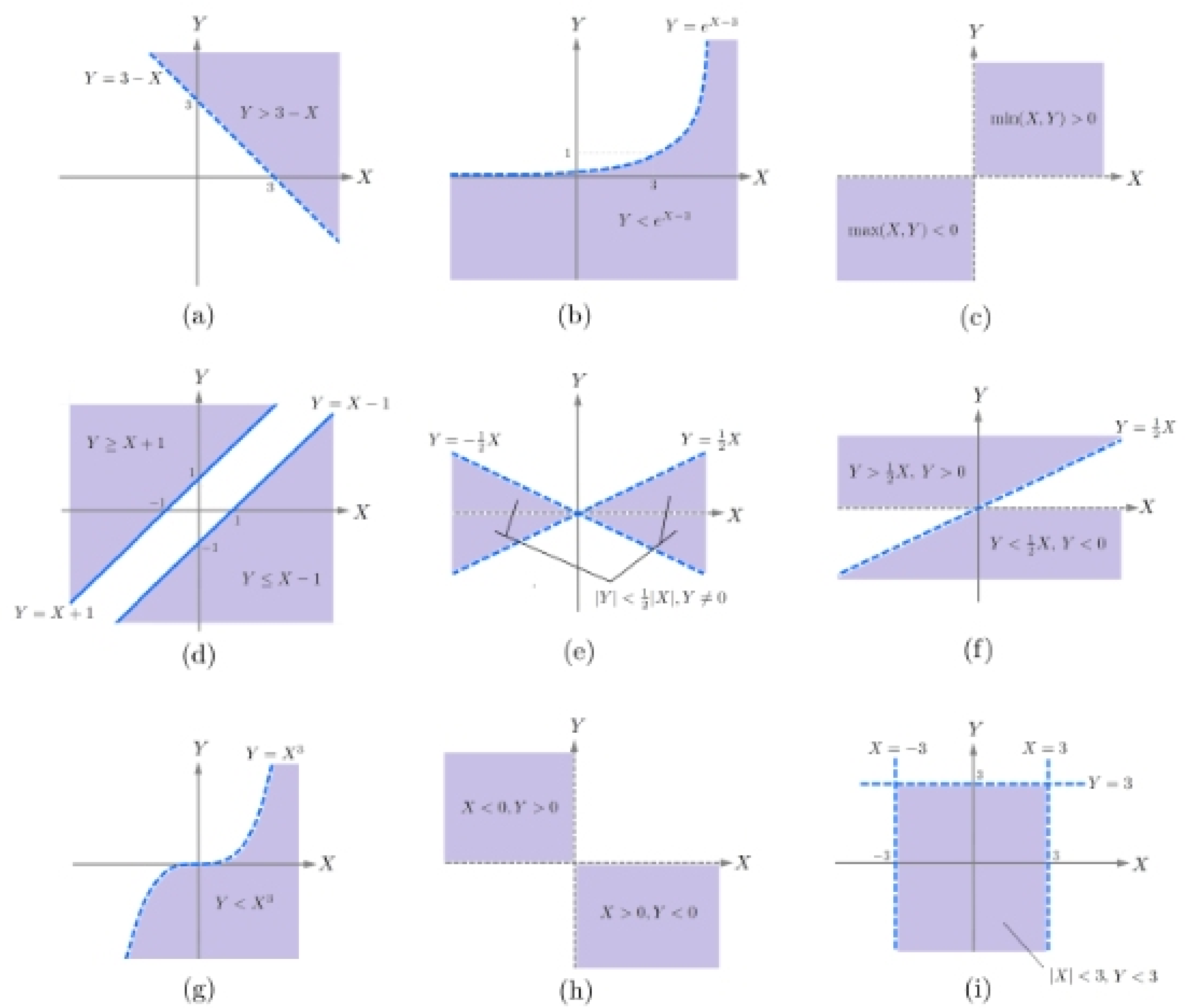
EE503 Homework Set 6 Solutions

Problem 6.1 (Text 5.8)

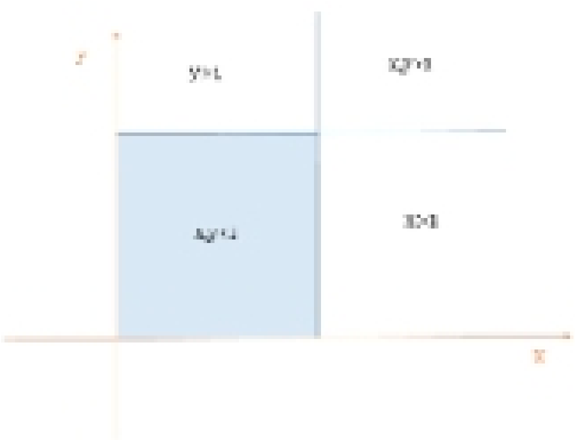


Problem 6.2 (Text 5.26)

(a) To find k we need to integrate the joint pdf over both variables such that the result equal to 1.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$
$$\int_0^1\int_0^1 k(x+y)dxdy = k\int_0^{\infty}(\frac{x^2}{2}+xy)\Big|_0^1 dy$$
$$= k\int_0^1\left(\frac{1}{2}+y\right)dy = k(\frac{1}{2}+\frac{y^2}{2})\Big|_0^1 = k \implies k=1$$

(b) Note that the cdf $F_{X,Y}(x,y) = P(X \le x, Y \le y)$ will depend on the value of x and y, for instance if the cdf should not depend on the value of x above 1. Thus to find the cdf let us integrate the pdf over each region.



- $0 \le x \le 1, 0 \le y \le 1$, note that we have to integrate the pdf over $(-\infty, x]$ and $(-\infty, y]$:

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\hat{x},\hat{y})d\hat{x}d\hat{y} = \int_0^y\int_0^x(\hat{x}+\hat{y})d\hat{x}d\hat{y} = \int_0^y(\frac{\hat{x}^2}{2}+\hat{x}\hat{y})\Big|_0^x d\hat{y}$$
$$= \int_0^y(\frac{x^2}{2}+x\hat{y})dy = (\frac{\hat{y}x^2}{2}+\frac{x\hat{y}^2}{2})\Big|_0^y = \frac{yx^2+xy^2}{2}$$

- $0 \le x \le 1, y > 1$, Same as above, but note that we do not care about the value of $y > 1$since the pdf is zero:

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\hat{x},\hat{y})d\hat{x}d\hat{y} = \int_0^1\int_0^x(\hat{x}+\hat{y})d\hat{x}d\hat{y} = \int_0^1(\frac{\hat{x}^2}{2}+\hat{x}\hat{y})\Big|_0^x d\hat{y}$$
$$= \int_0^y(\frac{x^2}{2}+x\hat{y})dy = (\frac{\hat{y}x^2}{2}+\frac{x\hat{y}^2}{2})\Big|_0^1$$
$$= \frac{x^2+x}{2}$$

- $x > 1, 0 \le y \le 1$, same as above we will have

$$F_{X,Y}(x,y) = \frac{y^2+y}{2}$$

- $x > 1, y > 1$,

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\hat{x},\hat{y})d\hat{x}d\hat{y} = \int_0^1\int_0^1(\hat{x}+\hat{y})d\hat{x}d\hat{y} = 1$$

- $x < 0$ OR $y < 0$

$$F_{X,Y}(x,y) = 0$$

Can you guess why? try to set a point on the figure eg. (-1,1), and see where the region $(X < -1, Y < 1)$. Does the pdf have non zero value there?

In summary, we have

$$F_{X,Y}(x,y) = \begin{cases} \frac{yx^2+xy^2}{2} & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ \frac{x^2+x}{2} & \text{if } 0 \le x \le 1, y > 1 \\ \frac{y^2+y}{2} & \text{if } x > 1, 0 \le y \le 1 \\ 1 & \text{if } x > 1, y > 1 \\ 0 & \text{otherwise} \end{cases}$$

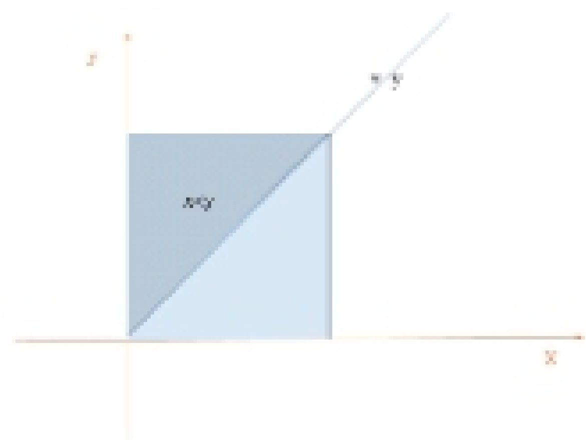
(c) The marginal pdf can be found easily :

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^1 (x+y)dy = x + \frac{1}{2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^1 (x+y)dx = \frac{1}{2} + y$$

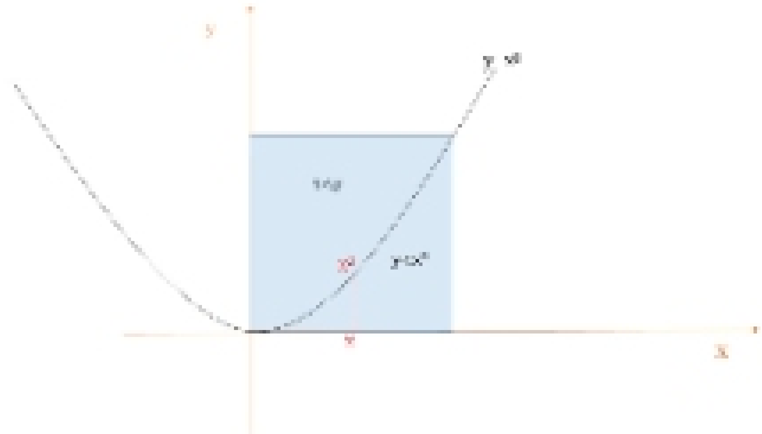
(d)

- $P(X < Y)$ we can plot a the region to help us



$$P(X < Y) = \int_0^1 \int_0^y (x+y)dxdy = \int_0^1 (\frac{x^2}{2} + xy)\Big|_0^y dy = \int_0^1 (\frac{3y^2}{2})dy = (\frac{y^3}{2})\Big|_0^1 = \frac{1}{2}$$

- $P(Y < X^2)$ again we can plot a the region to help us



$$P(Y < X^2) = \int_0^1 \int_0^{x^2} (x+y)dydx = \int_0^1 (\frac{x^2}{2} + xy)\Big|_0^{x^2} dx = \int_0^1 (\frac{x^4}{2} + x^3)dy$$
$$= (\frac{x^5}{10} + \frac{x^4}{4})\Big|_0^1 = \frac{1}{10} + \frac{1}{4} = 0.35$$