EE503 Homework Set 2 Solutions

Problem 2.1

We use a, b, and c to denote the name of Al, Bob and Chris respectively. We use their orders to denote the order they are drawn.
(a) $S = \{abc, acb, bac, cab, bca, cba\}$.
(b) $A = \{abc, acb\}$, $B = \{abc, cba\}$, $C = \{abc, bac\}$.
(c) $\{cab, bca\}$.
(d) $\{abc\}$.
(e) $\{abc, acb, bac, cba\}$.

Problem 2.2

$$\begin{aligned} Pr[B \cap C|A] &= \frac{Pr[B \cap C \cap A]}{Pr[A]} \\ &= \frac{Pr[\{abc\}]}{Pr[\{abc, acb\}]} \\ &= \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{3}} = \frac{1}{2}. \\ Pr[C|A \cap B] &= \frac{Pr[B \cap C \cap A]}{Pr[A \cap B]} \\ &= Pr[\{abc\}]/Pr[\{abc\}] = 1. \end{aligned}$$

Problem 2.3

(a) The total number of sequences is $2^8 = 256$.
(b) The probability is $1/256$.
(c) Let S be the event of success at second try and F be the event of fail at first try, then,

$$Pr[S|F] = 1/255.$$

which follows from the fact that you failed at the first try, which left you 255 possibilities for the second try. You randomly pick one number from 255 possibilities, which gives $1/255$.

Problem 2.4

$$\begin{aligned} Pr[A|B] &= \frac{Pr[A \cap B]}{Pr[B]} = \frac{Pr[\{x < 0\} \cap \{0 < x < 1\}]}{Pr[\{0 < x < 1\}]} = 0. \\ Pr[B|C] &= \frac{Pr[\{0 < x < 1\} \cap \{x > 0.75\}]}{Pr[x > 0.75]} = \frac{0.25/3}{1.25/3} = 1/5. \\ Pr[A|C'] &= \frac{Pr[\{x < 0\} \cap \{x \leq 0.75\}]}{Pr[\{x \leq 0.75\}]} = \frac{Pr[\{-1 \leq x < 0\}]}{Pr[\{-1 \leq x \leq 0.75\}]} = 1/1.75 = 4/7. \\ Pr[B|C'] &= \frac{Pr[\{0 < x < 1\} \cap \{x \leq 0.75\}]}{Pr[\{x \leq 0.75\}]} = 0.75/1.75 = 3/7. \end{aligned}$$

Problem 2.5

$a) Pr[A]Pr[B^c]Pr[C^c] + Pr[A^c]Pr[B]Pr[C^c] + Pr[A^c]Pr[B^c]Pr[C].$
$b) Pr[A]Pr[B]Pr[C^c] + Pr[A^c]Pr[B]Pr[C] + Pr[A]Pr[B^c]Pr[C].$
$c) 1 - Pr[A^c]Pr[B^c]Pr[C^c].$
$d) Pr[A]Pr[B]Pr[C^c] + Pr[A^c]Pr[B]Pr[C] + Pr[A]Pr[B^c]Pr[C] + Pr[A]Pr[B]Pr[C].$
$e) Pr[A^c]Pr[B^c]Pr[C^c].$

Problem 2.6

a) Since $Pr[F' \cap T'] = Pr[(F \cup T)'] = 0.25$, it follows

$$Pr[F \cup T] = 0.75.$$

Since $Pr[F] = 0.6$ and $Pr[T] = 0.7$, we have

$$Pr[F \cap T] = Pr[F] + Pr[T] - Pr[F \cup T] = 1.3 - 0.75 = 0.55.$$

Thus, the quantity asked in the question

$$Pr[\{F \cap T'\} \cup \{F' \cap T\}] = Pr[F \cup T] - Pr[F \cap T] = 0.75 - 0.55 = 0.2.$$

b)

$$Pr[T|F'] = \frac{Pr[T \cap F']}{Pr[F']} = \frac{Pr[T] - Pr[T \cap F]}{Pr[F']} = \frac{0.7 - 0.55}{0.4} = 0.375.$$

Problem 2.7

a)By inclusion-exclusion formula (or Venn diagram), we have

$$Pr[A + B] = Pr[A] + Pr[B] - Pr[A \cap B].$$

Since probability is nonnegative, $Pr[A \cap B] \geq 0$, which implies the claim. The equality holds when $Pr[A \cap B] = 0$, i.e. they are mutually exclusive or at least one of them has probability zero.

b) We reuse a) part 2 times, i.e.

$$\begin{aligned} Pr[A + B + C] &\leq Pr[A + B] + Pr[C] \\ &\leq Pr[A] + Pr[B] + Pr[C]. \end{aligned}$$

c) We prove this by induction. Base case $n = 1$, we have

$$Pr[A_1] = Pr[A_1],$$

and the formula holds.

Suppose the formula holds for $n = k$, then, we try to prove it also holds for $n = k + 1$. We have

$$\begin{aligned} Pr\left[\cup_{i=1}^{k+1} A_i\right] &= Pr\left[\left(\cup_{i=1}^{k} A_i\right) \cup A_{k+1}\right] \\ &\leq Pr\left[\cup_{i=1}^{k} A_i\right] + Pr[A_{k+1}] \\ &\leq \sum_{i=1}^{k} Pr[A_i] + Pr[A_{k+1}] = \sum_{i=1}^{k+1} Pr[A_i] \end{aligned}$$

where the first inequality uses our a) part and the second inequality implements our induction hypothesis that the formula holds for $n = k$. Overall, we finished our proof.

Note: For completeness, we provide the general form of *Inclusion-Exclusion formula* as a reference:

Inclusion-Exclusion Principle:
For events A_i, $i = 1, \ldots, m$, (not necessarily disjoint) we have:

$$P\left(\bigcup_{k=1}^{m} A_k\right) = \sum_{j=1}^{m} P(A_j) - \sum_{j<k} P(A_j \cap A_k) + \ldots + (-1)^{m+1} P\left(\bigcap_{k=1}^{m} A_k\right).$$

For $m = 2$:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

For $m = 3$:

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\ &\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\ &\quad + P(A_1 \cap A_2 \cap A_3). \end{aligned}$$