CSE310 HW01 Grading Keys

Please note that you have to typeset your assignment using either LaTeX or Microsoft Word. Hand-written assignment will not be graded. You need to submit a hardcopy before the lecture on the due date. You also need to submit an electronic version at the digital drop box. For the electronic version, you should name your file using the format HWxy-LastName-FirstName.

1. (10 pts) Let $f(n) = n^3$ and $g(n) = 100 \times n \times \log_2 n$. Find the smallest integer $N \geq 1$ such that $f(N) \leq g(N)$ but $f(N+1) > g(N+1)$. Show the values of N, $f(N)$, $g(N)$, $f(N+1)$ and $g(N+1)$.

   Solution:
   $N = 20$, $f(N) = 8000$, $g(N) = 8643.86$, $f(N+1) = 9261$, $g(N+1) = 9223.87$.

   Grading Keys:
   6 pts for N;
   1 pt for each of the other values.

2. (10 pts) For each function $f(n)$ (the row index in the following table) and time t (the column index in the following table), determine the largest size n of a problem that can be solved in time t, assuming that the algorithm takes $f(n)$ microseconds to solve an instance of a problem of size n. Fill the value n in the corresponding entry.

	1 second	1 minute	1 hour	1 day	1 year
$\sqrt{n}$	10^{12}	36×10^{14}	1296×10^{16}	746496×10^{16}	9945×10^{23}
n	10^6	6×10^7	36×10^8	864×10^8	31536×10^9
n^2	10^3	7745	6×10^4	293939	5615690
2^n	19	25	31	36	44

   Grading Keys:
   0.5 pt for each entry. If dicimal number is used, take off 1 pt from final score.

3. (10 pts) Prove that $\Theta(n) + O(n^{1.5}) \subseteq O(n^{1.5})$.
   Note that for this problem, you are proving that the set of functions on the left hand side (LHS)

is equal to the set of functions on the right hand side (RHS). The set on the LHS is algebraic sum of two sets (not the union): an element of the LHS has the form $f(n) = f_1(n) + f_2(n)$, where $f_1(n) \in \Theta(n)$ and $f_2(n) \in O(n^{1.5})$.

Solution: Because $f_1(n) \in \Theta(n)$, there exist positive constants c_1, c_2, N such that for any integer $n > N_1$, $c_1 n \leq f_1(n) \leq c_2 n$. Because $f_2(n) \in O(n^{1.5})$, there exist positive constants c_3, N_2 such that for any integer $n > N_2$, $f_2(n) \leq c_3 n^{1.5}$. Let $N_3 = \max\{N_1,\ N_2\}$. Thus, when $n > N_3$, $f(n) = f_1(n) + f_2(n) \leq c_2 n + c_3 n^{1.5}$. Let $c_4 = c_2 + c_3$, we thus have $f(n) \leq c_4 \times n^{1.5}$, $(n > N_3) \Rightarrow f(n) = O(n^{1.5})$. Therefore, $\Theta(n) + O(n^{1.5}) \subseteq O(n^{1.5})$.

Grading Keys:

2 pts for deriving the constants and inequality for $f_1(n)$;

2 pts for deriving the constants and inequality for $f_2(n)$;

2 pts for claiming c, N and deriving the inequality for $f(n)$;

4 pts for correctly deriving the constants and inequality for $f(n)$.

4. (10 pts) Prove that $n^{50} = O(2^n)$.

Solution: We have limit as follows:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n^{50}}{2^n} \\
&= \lim_{n\to\infty} \frac{(n^{50})'}{(2^n)'} \\
&= \lim_{n\to\infty} \frac{50n^{49}}{n2^{n-1}} = \lim_{n\to\infty} \frac{50n^{48}}{2^{n-1}} \\
&\vdots \\
&= \lim_{n\to\infty} \frac{50 \times 48 \times \cdots \times 2n}{2^{n-25}} = 0 < 1
\end{aligned}$$

Thus, there exists an integer N, such that for any integer $n > N$, $\frac{n^{50}}{2^n} < 1$. According to the definition of big-O, we have $n^{50} = O(2^n)$.

Grading Keys:

6 pts for claiming the existence of c and N without proving;

10 pts if the claim above is proved;

10 pts if they use $\lim_{n\to\infty} \frac{n^{50}}{2^n} = 0$.

5. (10 pts) Prove that $n^{n+3} \neq \Theta((n+3)^n)$.

Solution: We prove it by contradiction. Assume that $n^{n+3} = \Theta((n+3)^n)$. Thus, there exist positive constants c_1, c_2 and N_1 such that when $n > N_1$, $c_1(n+3)^n \leq n^{n+3} \leq c_2(n+3)^n$. However we also have the following derivation:

$$\lim_{n\to\infty} \frac{(n+3)^n}{n^{(n+3)}} = \lim_{n\to\infty} \frac{1}{n^3}(1+\frac{3}{n})^n = e^3 \times \lim_{n\to\infty} \frac{1}{n^3} = 0$$

This indicates that there exists a positive constant N_2, when $n > N_2$,

$$\begin{aligned} &\frac{(n+3)^n}{n^{n+3}} \leq \frac{1}{2c_2} < \frac{1}{c_2} \\ \Rightarrow &n^{n+3} > c_2(n+3)^n \end{aligned} \tag{1}$$

Inequality (1) contradicts with our assumption. Thus the claim holds.

Grading Keys:

2 pts for knowing to use proof by contradiction;

3 pts for specifying the false assumption;

5 pts for correct use of limit.