EE503 Solution to Quiz #6 10/8/2014

1. To get started, the math model for the function $g(z)$ is

$$g(z) = \begin{cases} z+1\,, & z \le -1 \\ 0\,, & -1 \le z \le 1 \\ z-1\,, & 1 \le z \end{cases}$$

(a)

$$\mathbb{E}\{y(u)\} = \int_{-\infty}^{\infty} g(z) f_x(z) dz = \frac{1}{2}\int_{-\infty}^{-1} (z+1)e^{z} dz + \frac{1}{2}\int_{1}^{\infty} (z-1)e^{-z} dz = 0$$

You can get this result by doing the integrals (which are finite), or by using the fact that the integrand is an odd function, i.e., $g(z)f_x(z) = -g(-z)f_x(-z)$ and no convergence issues exist.

(b) Using the fact that $\mathbb{E}\{y(u)\} = 0$ and the integrand is symmetric, i.e., $g^2(z)f_x(z) = g^2(-z)f_x(-z)$, the variance is given by the second moment

$$\begin{aligned} \mathbb{V}\{y(u)\} &= \mathbb{E}\{y^2(u)\} = \int_{-\infty}^{\infty} g^2(z) f_x(z) dz = \int_{1}^{\infty} (z-1)^2 e^{-z} dz \\ &= \int_{0}^{\infty} v^2 e^{-v-1} dv = e^{-1}\Gamma(3) = 2/e \end{aligned}$$

(c) This is simply the integral of the density function. So for $z < 0$,

$$F_x(z) = \int_{-\infty}^{z} f_x(z')dz' = \frac{1}{2}\int_{-\infty}^{z} e^{z'} dz' = \frac{1}{2}e^{z} \qquad \text{for } z < 0.$$

Because the density function $f_x(z')$ is symmetric, for $z > 0$,

$$1 - F_x(z) = \int_{z}^{\infty} f_x(z')dz' = \int_{-\infty}^{-z} f_x(z')dz' = \frac{1}{2}e^{-z} \qquad \text{for } z > 0.$$

Summarizing,

$$F_x(z) = \begin{cases} \frac{1}{2}e^{z}\,, & z \le 0 \\ 1 - \frac{1}{2}e^{-z}\,, & 0 \le z \end{cases}$$

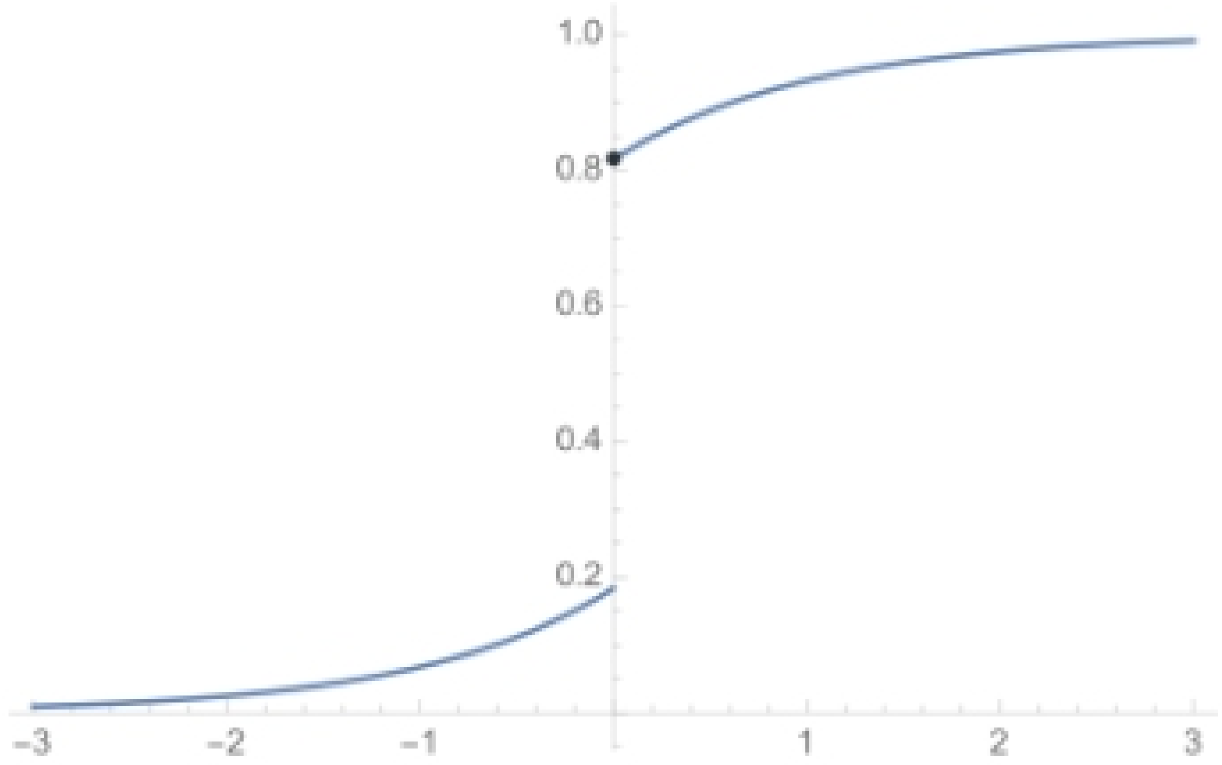
(d) For $z \ge 0$,

$$F_y(z) = \mathsf{P}\{u : y(u) \le z\} = \mathsf{P}\{u : x(u) \le z+1\} = F_x(z+1) = 1 - \frac{1}{2}e^{-z-1}$$

For $z < 0$,

$$F_y(z) = \mathsf{P}\{u : y(u) \le z\} = \mathsf{P}\{u : x(u) \le z-1\} = F_x(z-1) = \frac{1}{2}e^{z-1}$$

Here is a plot of $F_y(z)$. This function is right-continuous.


1.0
0.8
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-3
-2
-1
1
2
3