Math 132 Spring 2003, Exam 4 (Final Exam), <u>Solutions</u>

No calculators with a CAS are allowed. Part I, #1-17: Multiple Choice, 5 points/problem: mark your answers on the answer card.

1. What is the volume of the solid that results from revolving around the x-axis the region in the first quadrant under $y = \sqrt{\cos x}$ and above $0 \leq x \leq \frac{\pi}{2}$?

A) $\frac{1}{2}$ B) 1 C) $\frac{\pi}{2}$ **D)** π E) 2
F) $\frac{3\pi}{2}$ G) $\frac{3}{2}$ H) 2π I) 3 J) $\frac{\pi^2}{2}$

Using the disk method, $V = \int_0^{\pi/2} \pi f^2(x)\, dx = \int_0^{\pi/2} \pi(\sqrt{\cos x})^2\, dx = \int_0^{\pi/2} \pi \cos x\, dx = \pi \sin x|_0^{\pi/2} = \pi - 0 = \pi$.

2. A spring is stretched beyond its natural length. A meter is used to measure the force (lbs) needed when the spring has been stretched x ft. Some of the data collected is:

x (ft)	force (lbs)
0	0
1	$\frac{1}{2}$
2	1
3	$\frac{3}{2}$
4	2
5	$\frac{5}{2}$
6	3

How much work is done in stretching the spring 6 ft beyond its natural length?

A) 9 ft-lbs B) 10.5 ft-lbs C) 45.5 ft-lbs D) 18 ft-lbs
E) 4 ft-lbs F) 5.5 ft-lbs G) 7.5 ft-lbs H) 22 ft-lbs I) 32 ft-lbs
J) impossible to determine since there's no formula for the force $F(x)$ acting at x.

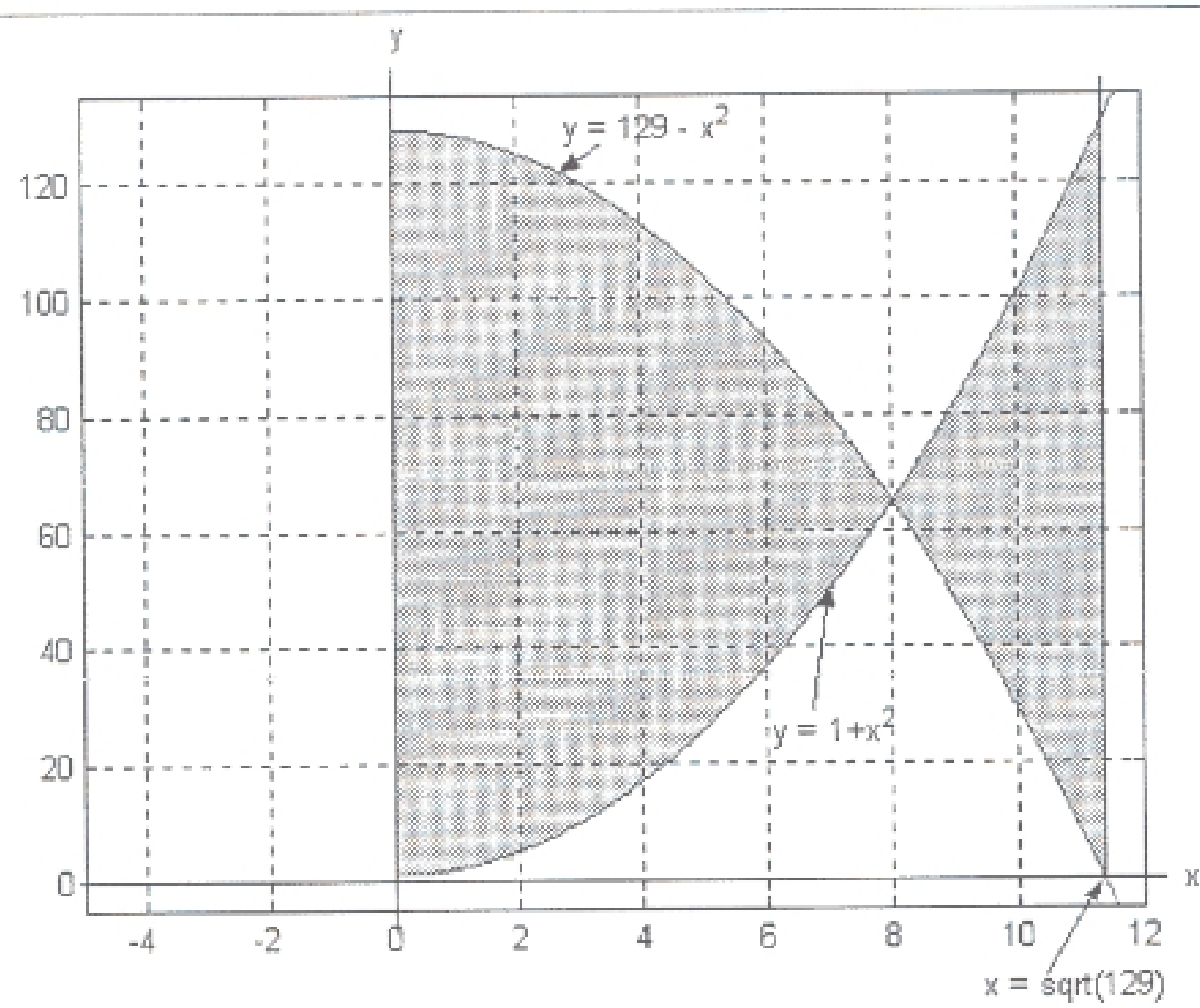
Work $W = \int_0^6 F(x)\, dx$ where $F(x)$ is the force required for x units of stretch. According to Hooke's Law, $F(x) = kx$ for some constant k. The data in the table shows that $k = \frac{1}{2}$. Therefore $W = \int_0^6 \frac{1}{2} x\, dx = \frac{x^2}{4}|_0^6 = \frac{36}{4} = 9$ (ft-lbs)

3. The area in the first quadrant between the curves $y = 1 + x^2$ and $y = 129 - x^2$ and above the interval $[0, \sqrt{129}]$ is given by which of the following ?

A) $\int_0^{\sqrt{129}} 2x^2 - 128\, dx$ B) $\int_0^{\sqrt{129}} 128 - 2x^2\, dx$

C) $\int_0^8 2x^2 - 128\, dx + \int_8^{\sqrt{129}} 128 - 2x^2\, dx$ D) $\int_0^6 2x^2 - 128\, dx + \int_6^{\sqrt{129}} 128 - 2x^2\, dx$

E) $\int_0^8 128 - 2x^2\, dx + \int_8^{\sqrt{129}} 2x^2 - 128\, dx$ F) $\int_0^6 128 - 2x^2\, dx + \int_6^{\sqrt{129}} 2x^2 - 128\, dx$



The region has two parts (shaded). The top and bottom boundary curves "switch" roles in the two regions. The curves intersect where $x = 8$. The area is given by

$\int_0^8 (129 - x^2) - (1 + x^2)\, dx + \int_8^{\sqrt{129}} (1 + x^2) - (129 - x^2)\, dx =$

$\int_0^8 (129 - x^2) - (1 + x^2)\, dx + \int_8^{\sqrt{129}} (1 + x^2) - (129 - x^2)\, dx =$

$\int_0^8 128 - 2x^2\, dx + \int_8^{\sqrt{129}} 2x^2 - 128\, dx.$

4. Find the value of $\int_2^\infty \frac{1}{x\ln x}\,dx$ (if the integral converges).

A) 0 B) 1 C) 2 D) 3 E) e
F) $\frac{e}{4}$ G) e^2 H) 5.429 I) 3.879 **J)** integral diverges

Substituting $u = \ln x$, $du = \frac{1}{x}dx$ gives
$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|\ln x| + C$. Therefore

$\int_2^\infty \frac{1}{x\ln x}dx = \lim_{t\to\infty} \ln|\ln x||_2^t = \infty$, so the integral diverges.

5. If y satisfies the differential equation $y' = \frac{x(y+1)}{x^2+3}$ and $y(1) = 3$, what is $y(0)$?

A) 0 B) $\frac{1}{2}$ C) 2 D) -1 E) $\sqrt{2}$
F) -2 **G)** $2\sqrt{3}-1$ H) 3 I) $\frac{\sqrt{2}}{3}$ J) $3\sqrt{2}$

We have $\frac{dy}{dx} = \frac{x(y+1)}{x^2+3}$. Separating variables gives $\frac{dy}{y+1} = \frac{x\,dx}{x^2+3}$, so $\int \frac{dy}{y+1} = \int \frac{x\,dx}{x^2+3}$.
Integrating both sides gives $\ln|y+1| = \frac{1}{2}\ln|x^2+3| + K = \ln(x^2+3)^{1/2} + K$. After exponentiating both sides we get

$$|y+1| = e^{\ln|x^2+3|^{1/2}+K} = e^K\sqrt{x^2+3} \text{ so that}$$
$$y+1 = \pm e^K\sqrt{x^2+3} = C\sqrt{x^2+3}.$$

Since $y = 3$ when $x = 1$, we get

$$4 = C\sqrt{4} \text{ so that } C = 2.$$

Therefore

$$y = 2\sqrt{x^2+3} - 1, \text{ and so } y(0) = 2\sqrt{3} - 1.$$