

Homework 7 - The Method of Partial Fractions

$$1) \frac{x+2}{(x^2+2)(x-1)^3(x^2-9)} = \frac{x+2}{(x^2+2)(x-1)^3(x+3)(x-3)}$$

$$= \frac{Ax+B}{x^2+2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} + \frac{F}{x+3} + \frac{G}{x-3}$$

$$2) \frac{2x+4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

$$2x+4 = A(x^2+4) + (Bx+C)(x-2)$$

$$\text{Substitute } x=2: 8 = 8A \rightarrow \boxed{A=1}$$

$$\text{Substitute } x=0: 4 = 4A - 2C$$

$$4 = 4 - 2C$$

$$\boxed{C=0}$$

$$2x+4 = Ax^2+4A + Bx^2+Cx-2Bx-2C$$

$$2x+4 = x^2(A+B) + x(C-2B) + 4A-2C$$

$$A+B=0 \rightarrow \boxed{B=-1}$$

$$3) \int \frac{dx}{(x-2)(x-4)}$$

$$\frac{1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}$$

$$1 = A(x-4) + B(x-2)$$

$$\text{Substitute } x=4: 1 = 2B \rightarrow B = \frac{1}{2}$$

$$\text{Substitute } x=2: 1 = -2A \rightarrow A = -\frac{1}{2}$$

$$\int \frac{dx}{(x-2)(x-4)} = \int \left[-\frac{1}{2} \left(\frac{1}{x-2} \right) + \frac{1}{2} \left(\frac{1}{x-4} \right) \right] dx$$

$$= -\frac{1}{2} \ln|x-2| + \frac{1}{2} \ln|x-4| + C$$

$$= -\ln\sqrt{|x-2|} + \ln\sqrt{|x-4|} + C$$

$$= \boxed{\ln\sqrt{\left| \frac{x-4}{x-2} \right|} + C}$$

$$4) \int \frac{x^2}{x^2+9} dx$$

$$x^2+9 \overline{) \begin{array}{r} x^2 \\ x^2+9 \\ \hline -9 \end{array}}$$

$$\frac{x^2}{x^2+9} = 1 - \frac{9}{x^2+9}$$

$$\int \frac{x^2}{x^2+9} dx = \int \left(1 - \frac{9}{x^2+9} \right) dx$$

$$= x - \frac{9}{3} \tan^{-1} \frac{x}{3} + C$$

$$= \boxed{x - 3 \tan^{-1} \frac{x}{3} + C}$$

$$5) \int \frac{(x^2+11x)}{(x-1)(x+1)^2} dx$$

$$\frac{x^2+11x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2+11x = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

Substitute $x=1$: $12 = 4A \rightarrow A=3$

Substitute $x=-1$: $-10 = -2C \rightarrow C=5$

Substitute $x=0$: $\rightarrow 0 = A - B - C$

$$\int \frac{x^2+11x}{(x-1)(x+1)^2} dx = \int \left[\frac{3}{x-1} - \frac{2}{x+1} + \frac{5}{(x+1)^2} \right] dx$$

$$= \boxed{3 \ln|x-1| - 2 \ln|x+1| - \frac{5}{x+1} + C}$$

$$6) \int \left(\frac{x^2-x+1}{x^2+x} \right) dx$$

$$\begin{array}{r} x^2+x \overline{) x^2-x+1} \\ \underline{x^2+x} \\ -2x+1 \end{array}$$

$$\int \frac{x^2-x+1}{x^2+x} dx = \int \left(1 + \frac{-2x+1}{x(x+1)} \right) dx$$

$$\frac{-2x+1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$-2x+1 = A(x+1) + B(x)$$

$$x=-1 \rightarrow 3 = -B \rightarrow B = -3$$

$$x=0 \rightarrow 1 = A$$

$$\int \left(1 + \frac{1}{x} - \frac{3}{x+1} \right) dx = \boxed{x + \ln|x| - 3 \ln|x+1| + C}$$