Homework 6 - Trigonometric Substitution

1) $\int \sqrt{16-x^2}\,dx$

$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$= \int \sqrt{16-16\sin^2\theta}\,(4\cos\theta)\,d\theta$

$= \int \sqrt{16\cos^2\theta}\,(4\cos\theta)\,d\theta$

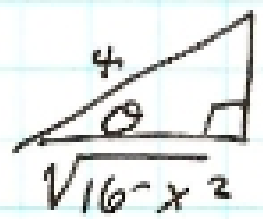
$= \int 4\cos\theta\,(4\cos\theta)\,d\theta$

$= \int 16\cos^2\theta\,d\theta$

$= \int 8\,(1+\cos 2\theta)\,d\theta$

$= 8\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

$\theta = \sin^{-1}\frac{x}{4}$



$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{x}{4}\right)\left(\sqrt{16-x^2}\right)$

$= \boxed{8\left(\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{16-x^2}\right) + C}$

2) $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$

$x = \sin\theta \quad dx = \cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)$

$= \frac{\pi}{12} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)$

$= \boxed{\frac{\pi}{12} - \frac{\sqrt{3}}{8}}$

3) $\int \frac{dx}{x\sqrt{x^2+16}}$

$x = 4\tan\theta \quad dx = 4\sec^2\theta\, d\theta$

$= \int \frac{4\sec^2\theta}{4\tan\theta\sqrt{16\tan^2\theta+16}}\, d\theta$

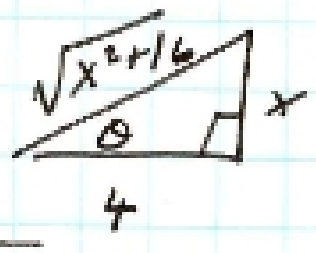
$= \int \frac{\sec^2\theta}{\tan\theta(4\sec\theta)}\, d\theta$

$= \int \frac{1}{4}\frac{\sec\theta}{\tan\theta}\, d\theta$

$= \int \frac{1}{4}\csc\theta\, d\theta$

$= \frac{1}{4}\ln|\csc\theta - \cot\theta| + C$

$\tan\theta = \frac{x}{4}$



$\csc\theta = \frac{\sqrt{x^2+16}}{x}$

$\cot\theta = \frac{4}{x}$

$= \boxed{\frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}}{x} - \frac{4}{x}\right| + C}$

OR

$\boxed{\frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}-4}{x}\right| + C}$