Homework 10 - Solutions

1) $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y = 3x+1$

$\frac{dy}{dx} = 3$

$S = \int_0^3 \sqrt{1+3^2}\,dx$

$= \sqrt{10}\,x\Big|_0^3$

$= \boxed{3\sqrt{10}}$

2) $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y = \frac{1}{12}x^3 + x^{-1}$

$\frac{dy}{dx} = \frac{3}{12}x^2 - 1^{-2} = \frac{1}{4}x^2 - x^{-2}$

$\left(\frac{dy}{dx}\right)^2 = \frac{x^4}{16} - 2\left(\frac{1}{4}x^2\right)\left(x^{-2}\right) + x^{-4}$

$= \frac{x^4}{16} - \frac{1}{2} + x^{-4}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{x^4}{16} + \frac{1}{2} + x^{-4}$

$= \left(\frac{x^2}{4} + x^{-2}\right)^2$

$S = \int_1^2 \sqrt{\left(\frac{x^2}{4}+x^{-2}\right)^2}\,dx$

$= \int_1^2 \left(\frac{x^2}{4}+x^{-2}\right)dx$

$= \left(\frac{x^3}{12} - x^{-1}\right)\Big|_1^2$

$= \frac{8}{12} - \frac{1}{2} - \frac{1}{12} + 1 = \boxed{\frac{13}{12}}$

3) $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y = e^x$

$\frac{dy}{dx} = e^x$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+e^{2x}}$

$S = \int_0^{\frac{1}{2}} \sqrt{1+e^{2x}}\,dx$

$u = \sqrt{1+e^{2x}}$

$u^2 = 1+e^{2x}$; $e^{2x} = u^2-1$

$2u\,du = 2e^{2x}\,dx$

$dx = \frac{u}{e^{2x}}\,du$

$dx = \frac{u}{u^2-1}\,du$

$S = \int_{\sqrt{2}}^{\sqrt{1+e}} u\left(\frac{u}{u^2-1}\right)du$

$= \int_{\sqrt{2}}^{\sqrt{1+e}} \frac{u^2}{u^2-1}\,du$

$u^2-1 \overline{)\,u^2\;\;1}$

$\frac{u^2-1}{1}$

$\frac{u^2}{u^2-1} = 1 + \frac{1}{u^2-1}$

$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$1 = A(u+1) + B(u-1)$

$u=-1 \rightarrow 1 = -2B$, $B = -\frac{1}{2}$

$u=1 \rightarrow 1 = 2A$, $A = \frac{1}{2}$

$S = \int_{\sqrt{2}}^{\sqrt{1+e}} \left[1 + \frac{1}{2}\left(\frac{1}{u-1}\right) - \frac{1}{2}\left(\frac{1}{u+1}\right)\right]du$

$$S = \left[u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| \right]_{\sqrt{2}}^{\sqrt{1+e}}$$

$$S = \sqrt{1+e} + \frac{1}{2}\ln\left|\sqrt{1+e}-1\right| - \frac{1}{2}\ln\left|\sqrt{1+e}+1\right|$$
$$- \sqrt{2} - \frac{1}{2}\ln\left|\sqrt{2}-1\right| + \frac{1}{2}\ln\left|\sqrt{2}+1\right|$$

$$\boxed{S = \sqrt{1+e} - \sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+e}-1}{\sqrt{1+e}+1}\right) + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)}$$