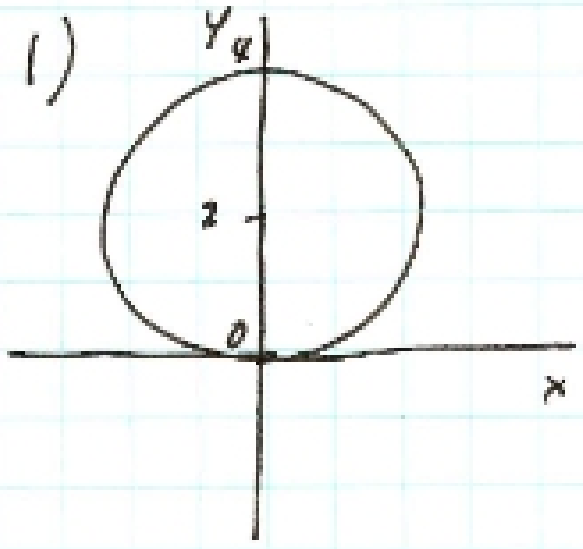
Homework 30 - Area and Arc Length in Polar Coordinates

1)



$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

$$A = \int_0^\pi \frac{1}{2} (4\sin\theta)^2 \, d\theta$$

$$= \int_0^\pi 8\sin^2\theta \, d\theta$$

$$= \int_0^\pi 4(1-\cos 2\theta) \, d\theta$$

$$= 4\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^\pi$$

$$= \boxed{4\pi}$$

2) $$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(4\sec\left(\theta - \tfrac{\pi}{4}\right)\right)^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} 8\sec^2\left(\theta - \tfrac{\pi}{4}\right) d\theta$$

$$= 8\tan\left(\theta - \tfrac{\pi}{4}\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 8\left(\tan\left(\tfrac{\pi}{4}\right) - \tan\left(-\tfrac{\pi}{4}\right)\right)$$

$$= 8(1+1)$$

$$= \boxed{16}$$

3) a) $A = \int_0^{2\pi} \frac{1}{2}(1-\cos\theta)^2 \, d\theta$

$= \int_0^{2\pi} \frac{1}{2}(1-2\cos\theta+\cos^2\theta) \, d\theta$

$= \int_0^{2\pi} \frac{1}{2}(1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)) \, d\theta$

$= \int_0^{2\pi} (\frac{3}{4}-\cos\theta+\frac{1}{4}\cos 2\theta) \, d\theta$

$= \left(\frac{3}{4}\theta - \sin\theta + \frac{1}{8}\sin 2\theta\right)\Big|_0^{2\pi}$

$= \frac{3}{4}(2\pi)$

$= \boxed{\frac{3\pi}{2}}$

b) $A = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos\theta)^2 \, d\theta$

$= \left(\frac{3}{4}\theta - \sin\theta + \frac{1}{8}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{3}{4}\left(\frac{\pi}{2}\right) - 1$

$= \boxed{\frac{3\pi}{8} - 1}$

4) a) intersection : $1 = \sqrt{2}\cos\theta$

$$\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = -\frac{\pi}{4}, \frac{\pi}{4}$$

$$A_A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\left[(\sqrt{2}\cos\theta)^2 - 1^2\right] d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}\left[2\cos^2\theta - 1\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[1 + \cos 2\theta - 1\right] d\theta$$

$$= \left(\frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}$$

$$A_A = \boxed{\frac{1}{2}}$$

b) A_B = (Area within $r = \sqrt{2}\cos\theta$) $- A_A$

$r = \sqrt{2}\cos\theta$ is a circle with radius $= \frac{\sqrt{2}}{2}$

$$A_{circle} = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}\pi$$

$$A_B = \frac{\pi}{2} - \frac{1}{2}$$

$$= \boxed{\frac{\pi}{2} - \frac{1}{2}}$$

Another option: